Approximation of a circular cylindrical shell by Clough-Johnson flat plate finite elements

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APPROXIMATON OF A CIRCULAR CYLINDRICAL SHELL

BY CLOUGH-JOHNSON FLAT PLATE FINITE ELEMENTS.

Michel BERNADOU ([1]) - Yves DUCATEL ([2])

Résumé : Dans ce rapport, nous étudions l'approximation d'une coque cylindrique circulaire par une méthode non conforme utilisant les éléments finis plats de CLOUGH-JOHNSON. Nous donnons les conditions de compatibilité qui doivent être satisfaites par les degrés de liberté en chaque noeud de la triangulation. Puis, nous montrons que la convergence est assurée pour certaines familles particulières de triangulation qui, en pratique, sont faciles à mettre en oeuvre.

Abstract : In this report, we study the approximation of a right circular cylindrical shell by a nonconforming method using CLOUGH-JOHNSON flat plate finite elements. Compatibility conditions which have to be satisfied by the degrees of freedom at every node of the triangulation are given. Then, we prove that convergence is insured for particular families of triangulations which are practically easy to implement.

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- INTRODUCTION -

The purpose of this paper is to extend to the case of a portion of a right circular cylinder the studies of CIARLET [1] and C. JOHNSON [1] concerning the particular case of a circular arch and the one of BERNADOU-DUCATEL [1] concerning the case of a general arch.

In order to give an approximation of the deformation problem of a portion of a right circular cylinder, we intend to analyze a nonconforming finite element method using CLOUGH-JOHNSON flat plate elements. So we introduce :

- a *nonconforming approximation of the geometry* of the considered shell using facet elements ;
- a *pseudo-conforming approximation of the components of the displacement*, i.e. an approximation using conforming plate elements over every flat element. Then, the connection between degrees of freedom attached to a same vertex of the triangulation is obtained through the *compatibility conditions*. These conditions insure the consistency between exact and approximated bilinear and linear forms and they are crucial in proving the convergence of this method.

Particularly, we emphasize that the convergence of the method is insured provided the families of triangulations are constructed so that any triangle have one side parallel to the cylinder axis.

Besides, we have to mention other alternatives : ARGYRIS-DUNNE-MALEJANNAKIS-SCHELKLE [1] describe a facet triangular plate and shell finite element whose 18 degrees of freedom are the displacement and the rotation at each corner ; this element is applicable to thin or thick shells with or without transverse shear deformation. OLSON-BEARDEN [1] reformulate the previous 18 degrees of freedom element and introduce constraints to reduce these to 9. DAWE [1] proposes a facet element whose degrees of freedom are the displacement u at each corner point and the normal slope $\frac{\partial u}{\partial n}$ at the mid-points of each side. Finally, let us notice the alternative of IRONS [1] with his semi-loof shell element.

- CONTENTS -

1 - THE CONTINUOUS PROBLEM

In this paragraph, we give the formulation of the continuous problem deduced as a particular case from the general linear shell model of W.T. KOITER [1].

1.1 - Definition of the middle surface of a right circular cylinder

Let $\Omega =]\underline{\xi}^1,\bar{\xi}^1[\times]-H,+H[$ be an open bounded subset of a plane $\mathscr{E}^2$, with $0 \le \underline{\xi}^1 < \bar{\xi}^1 \le 2\pi$, $0 < H$.

Let $\mathscr{E}^3$ be the usual Euclidean space. We denote by $(0,\vec{e}_1,\vec{e}_2,\vec{e}_3)$ an orthonormal reference system of $\mathscr{E}^3$.

The middle surface $\bar{\mathscr{S}}$ of the portion of a right circular cylinder (cf. Figure 1.1-1), is the image in $\mathscr{E}^3$ of the set $\bar{\Omega}$ by the mapping

(1.1-1) $$\vec{\phi}(\xi^1,\xi^2) = R \cos \xi^1 \,\vec{e}_1 + R \sin \xi^1 \,\vec{e}_2 + \xi^2 \,\vec{e}_3 \ .$$

By convention, we shall use subsequently Greek letters (resp. Latin letters) for indices which take their values in the set $\{1,2\}$ (resp. $\{1,2,3\}$). For these indices, we shall use Einstein' convention for summation. Finally, the notations $f,_\alpha$, $f,_{\alpha\beta}$, etc..., will denote the partial derivatives $\dfrac{\partial f}{\partial \xi^\alpha}$, $\dfrac{\partial^2 f}{\partial \xi^\alpha \partial \xi^\beta}$, etc...

As usual, the *covariant basis* is defined by

(1.1-2) $$\vec{a}_1 = \frac{\partial \vec{\phi}}{\partial \xi^1} = \begin{cases} -R \sin \xi^1 \\ R \cos \xi^1 \\ 0 \end{cases} , \quad \vec{a}_2 = \frac{\partial \vec{\phi}}{\partial \xi^2} = \begin{cases} 0 \\ 0 \\ 1 \end{cases} , \quad \vec{a}_3 = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|} = \begin{cases} \cos \xi^1 \\ \sin \xi^1 \\ 0 \end{cases}$$

The *first fundamental form* of the surface $\mathscr{S}$ is given by

(1.1-3) $$a_{11} = R^2 \ , \quad a_{12} = a_{21} = 0 \ , \quad a_{22} = 1 \ .$$

We denote

(1.1-4) $$a = a_{11}a_{22} - (a_{12})^2 = R^2 \ .$$

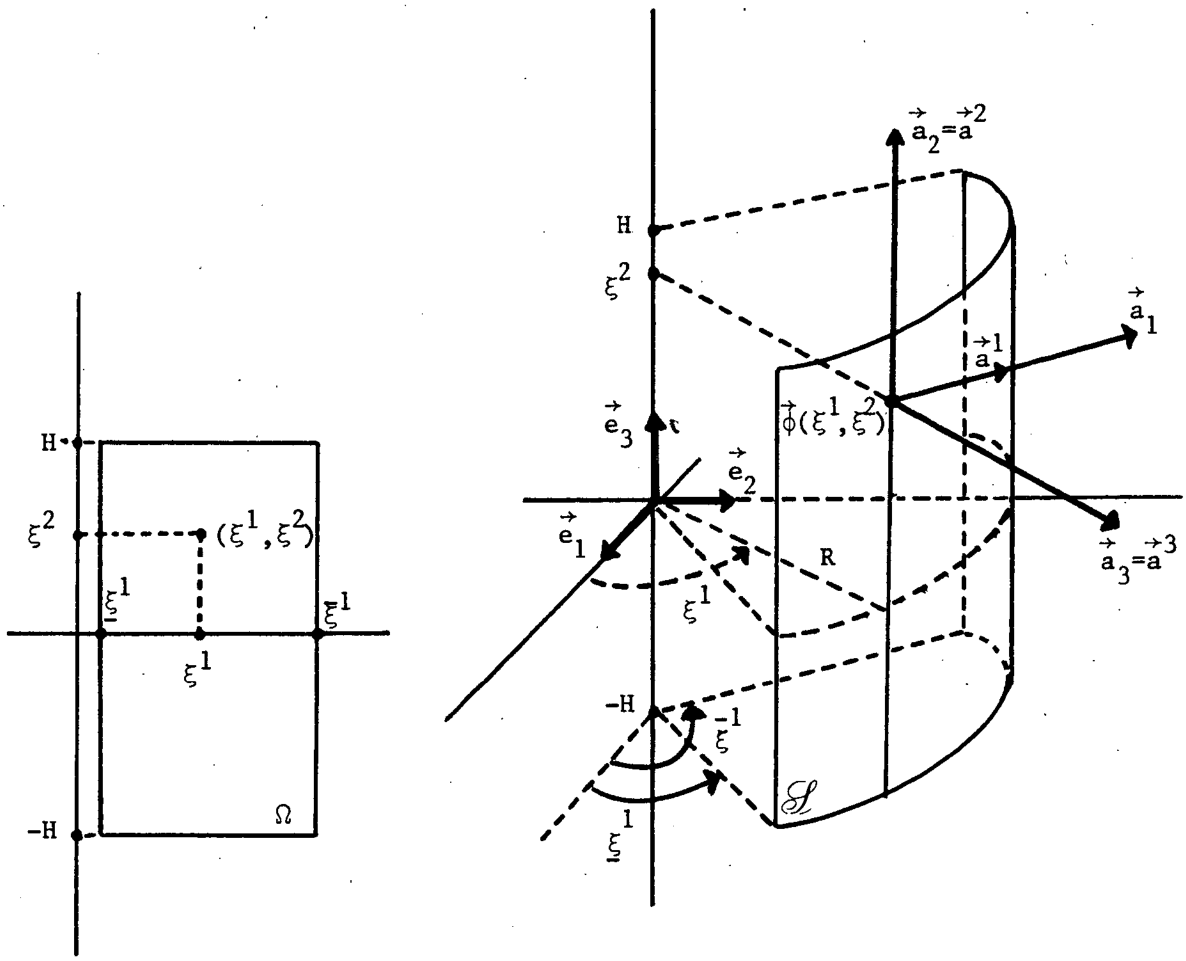


Figure 1.1-1 : Definition of the middle surface of the shell

The matrix $(a^{\alpha\beta})$, inverse of the matrix $(a_{\alpha\beta})$, is such that

$$a^{11} = \frac{1}{R^2} \quad , \quad a^{12} = a^{21} = 0 \quad , \quad a^{22} = 1 \ . \tag{1.1-5}$$

Then, to the covariant basis $(\vec{a}_i)$, we associate the *contravariant basis* $(\vec{a}^i)$ of the middle surface $\mathscr{S}$ by $\vec{a}^\alpha = a^{\alpha\beta}\vec{a}_\beta$, $\vec{a}^3 = \vec{a}_3$, i.e.,

$$\vec{a}^1 = \frac{1}{R}\begin{pmatrix} -\sin\xi^1 \\ \cos\xi^1 \\ 0 \end{pmatrix} \quad , \quad \vec{a}^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \vec{a}^3 = \vec{a}_3 \ . \tag{1.1-6}$$

The *second fundamental form* $(b_{\alpha\beta})$ of the middle surface $\mathscr{S}$ is defined by $b_{\alpha\beta} = b_{\beta\alpha} = -\vec{a}_\alpha \cdot \vec{a}_{3,\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta}$, i.e.,

$$b_{11} = -R \ , \ b_{12} = b_{21} = b_{22} = 0 \ . \tag{1.1-7}$$

Since $b^\beta_\alpha = a^{\beta\lambda} b_{\lambda\alpha}$, then

$$b^1_1 = -\frac{1}{R} \quad , \quad b^1_2 = b^2_1 = b^2_2 = 0 \ . \tag{1.1-8}$$

The *unknowns of the problem* are the three functions

$$u_i : (\xi^1,\xi^2) \in \bar{\Omega} \longrightarrow u_i(\xi^1,\xi^2) \in \mathbb{R} \quad , \quad i = 1,2,3 \ ,$$

which represent the covariant components of the displacement $\vec{u} = \vec{u}(\xi^1,\xi^2)$ of the point $\vec{\phi}(\xi^1,\xi^2)$, i.e., $\vec{u} = \sum_{i=1}^{3} u_i \vec{a}^i$.

To any displacement field $\vec{v} = v_i \vec{a}^i$, we associate the *strain tensor* $(\gamma_{\alpha\beta})$ and the *change of curvature tensor* $(\bar{\rho}_{\alpha\beta})$ through the relations

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2}(v_{\beta|\alpha} + v_{\alpha|\beta}) - b_{\alpha\beta} v_3 \ , \tag{1.1-9}$$

$$\bar{\rho}_{\alpha\beta}(\vec{v}) = v_{3|\alpha\beta} - b^\lambda_\alpha b_{\lambda\beta} v_3 + b^\lambda_{\beta|\alpha} v_\lambda + b^\lambda_\beta v_{\lambda|\alpha} + b^\lambda_\alpha v_{\lambda|\beta} \ , \tag{1.1-10}$$

where the covariant derivatives are given by

$$(1.1\text{-}11)\quad \begin{cases} v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}\, v_{\lambda} \quad , \quad v_{3|\alpha} = v_{3,\alpha} \ , \\ v_{3|\alpha\beta} = v_{3,\alpha\beta} - \Gamma^{\lambda}_{\alpha\beta}\, v_{3,\lambda} \ , \\ b^{\lambda}_{\beta|\alpha} = b^{\lambda}_{\beta,\alpha} + \Gamma^{\lambda}_{\alpha\nu}\, b^{\nu}_{\beta} - \Gamma^{\nu}_{\beta\alpha}\, b^{\lambda}_{\nu} \ , \end{cases}$$

with

$$(1.1\text{-}12)\quad \Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha} = \vec{a}^{\lambda} \cdot \vec{a}_{\alpha,\beta} \quad \text{(Christoffel symbols).}$$

In his analysis, KOITER [1] proposes the *modified change of curvature tensor* $\rho_{\alpha\beta}$

$$(1.1\text{-}13)\quad \rho_{\alpha\beta} = \bar{\rho}_{\alpha\beta} - \frac{1}{2}\,(b^{\nu}_{\alpha}\,\gamma_{\nu\beta} + b^{\nu}_{\beta}\,\gamma_{\nu\alpha})$$

which introduces in the energy integral an approximation of the same order than the one obtained from the tensor $\bar{\rho}_{\alpha\beta}$ and which will be more convenient for our analysis, particularly for the proof of the convergence of the approximate problem.

More precisely, we have for the right circular cylinder :

$$(1.1\text{-}14)\quad \Gamma^{\lambda}_{\alpha\beta} = 0$$

$$(1.1\text{-}15)\quad \begin{cases} \gamma_{11}(\vec{v}) = v_{1,1} + R\, v_3 \\ \gamma_{12}(\vec{v}) = \frac{1}{2}\,(v_{1,2} + v_{2,1}) \\ \gamma_{22}(\vec{v}) = v_{2,2} \end{cases}$$

$$(1.1\text{-}16)\quad \begin{cases} \rho_{11}(\vec{v}) = v_{3,11} - \frac{1}{R}\, v_{1,1} \\ \rho_{12}(\vec{v}) = v_{3,12} - \frac{1}{4R}\,(3v_{1,2} - v_{2,1}) \\ \rho_{22}(\vec{v}) = v_{3,22} \ . \end{cases}$$

By using relations

$$\gamma^{\alpha}_{\beta}(\vec{v}) = a^{\alpha\lambda}\,\gamma_{\lambda\beta}(\vec{v}) \quad , \quad \rho^{\alpha}_{\beta}(\vec{v}) = a^{\alpha\lambda}\,\rho_{\lambda\beta}(\vec{v}) \ , \tag{1.1-17}$$

we derive :

$$\left\{\begin{aligned} \gamma^1_1(\vec{v}) &= \frac{1}{R^2}\,(v_{1,1} + R\,v_3) \\ \gamma^1_2(\vec{v}) &= \frac{1}{2R^2}\,(v_{1,2} + v_{2,1}) \\ \gamma^2_1(\vec{v}) &= \frac{1}{2}\,(v_{1,2} + v_{2,1}) \\ \gamma^2_2(\vec{v}) &= v_{2,2} \end{aligned}\right. \tag{1.1-18}$$

and

$$\left\{\begin{aligned} \rho^1_1(\vec{v}) &= \frac{1}{R^2}\,(v_{3,11} - \frac{1}{R}\,v_{1,1}) \\ \rho^1_2(\vec{v}) &= \frac{1}{R^2}\,\{v_{3,12} - \frac{1}{4R}\,(3v_{1,2} - v_{2,1})\} \\ \rho^2_1(\vec{v}) &= v_{3,12} - \frac{1}{4R}\,(3v_{1,2} - v_{2,1}) \\ \rho^2_2(\vec{v}) &= v_{3,22} \end{aligned}\right. \tag{1.1-19}$$

1.2 - Variational formulation of the continuous problem

Let Γ_o be a measurable part of the boundary Γ such that

$$\text{meas}(\Gamma_o) > 0 \ .$$

Subsequently, we assume that the shell is *clamped* on the part $\partial\mathscr{S}_o = \vec{\phi}(\Gamma_o)$ of the middle surface boundary, i.e.,

$$\vec{u}|_{\Gamma_o} = \vec{0} \quad , \quad \partial_n u_3|_{\Gamma_o} = 0 \quad ,$$

where ∂_n denotes the outer normal derivative operator.

Then, the *admissible displacement space* $\vec{V}$ is defined by

(1.2-1) $$\vec{V} = \{\vec{v} = (v_1,v_2,v_3) \in (H^1(\Omega))^2 \times H^2(\Omega) \ ; \ \vec{v}|_{\Gamma_o} = \vec{0} \ , \ \partial_n v_3|_{\Gamma_o} = 0\} \ .$$

Equipped with the scalar product

$$((\vec{u},\vec{v})) = \sum_{\alpha=1}^{2} ((u_\alpha,v_\alpha))_{1,\Omega} + ((u_3,v_3))_{2,\Omega} \ ,$$

the space $\vec{V}$ is a Hilbert space, the corresponding norm being denoted by

(1.2-2) $$\|\vec{v}\| = [((\vec{v},\vec{v}))]^{1/2} \ .$$

For simplicity and without loss of generality, we assume the shell be elastic, homogeneous and isotropic. Then, using the definition (1.1-13) of the change of curvature tensor, the *strain energy of the shell* can be written as

(1.2-3) $$\left\{ \begin{aligned} a(\vec{u},\vec{v}) = \int_\Omega \frac{Ee}{1-\nu^2} \{ & (1-\nu) \ \gamma^\alpha_\beta(\vec{u}) \ \gamma^\beta_\alpha(\vec{v}) + \nu \ \gamma^\alpha_\alpha(\vec{u}) \ \gamma^\beta_\beta(\vec{v}) + \\ & + \frac{e^2}{12} [(1-\nu) \ \rho^\alpha_\beta(\vec{u}) \ \rho^\beta_\alpha(\vec{v}) + \nu \ \rho^\alpha_\alpha(\vec{u}) \ \rho^\beta_\beta(\vec{v})] \sqrt{a} \ d\xi^1 \ d\xi^2 \end{aligned} \right.$$

where e, E, ν denote respectively the thickness, the Young modulus and the Poisson coefficient of the shell and where the explicit expressions of tensors (γ^α_β) and (ρ^α_β) are given by (1.1-18) and (1.1-19).

The *potential energy of exterior forces* associated to a displacement $\vec{v} = v_i \ \vec{a}^i$ of the particles of the middle surface $\mathcal{S}$ is

(1.2-4) $$f(\vec{v}) = \int_\Omega \vec{p}\cdot\vec{v} \ \sqrt{a} \ d\xi^1 \ d\xi^2 = \int_\Omega p^i \ v_i \ \sqrt{a} \ d\xi^1 \ d\xi^2$$

where the functions p^i are the contravariant components over the basis $(\vec{a}_i)$ of the density per unit surface of the resultant over the middle surface of exterior forces applied to the shell. This expression corresponds to the case of a shell which is *free* on the complementary part $\Gamma_1 = \Gamma - \Gamma_o$ of the boundary Γ.

Then, the problem of deformation of the portion $\mathscr{S}$ of right circular cylinder admits the following variational formulation :

$$(1.2\text{-}5)\quad \begin{cases} \underline{\text{Find}}\ \vec{u} \in \vec{V}\ \underline{\text{such that}} \\ a(\vec{u},\vec{v}) = f(\vec{v}) \quad , \quad \forall \vec{v} \in \vec{V} \quad , \end{cases}$$

where the forms $a(\cdot,\cdot)$ and $f(\cdot)$ are defined by (1.2-3) and (1.2-4) respectively.

1.3 - Existence and uniqueness theorem

Theorem 1.3-1 : The problem (1.2-5) has one and only one solution.

Proof : see BERNADOU-CIARLET [1].

□

2 - A CONFORMING FINITE ELEMENT METHOD

In this paragraph, we consider an approximation of the displacement components using a conforming finite element method. This stage will be an useful intermediary for the analysis of a flat plate element approximation.

2.1 - The finite element space $\vec{V}_h$

Since the set $\bar{\Omega}$ is rectangular, we may exactly cover $\bar{\Omega}$ by *regular* families of triangulations $\mathscr{C}_h$ in the following sense :

(i) there exists a constant σ such that

$$(2.1\text{-}1)\qquad \forall K \in \bigcup_h \mathscr{C}_h \quad , \quad \frac{h_K}{\rho_K} \leq \sigma \quad ,$$

where h_K = diam(K) and ρ_K = sup {diam(S), S is a ball contained in K} .

(ii) the quantity

(2.1-2) $$h = \max_{K \in \mathcal{T}_h} h_K$$

approaches zero.

To each triangulation $\mathcal{T}_h$, we associate a product of finite element spaces $\vec{X}_h = X_{h1} \times X_{h1} \times X_{h2}$. Next, we define a subspace $\vec{V}_h = V_{h1} \times V_{h1} \times V_{h2}$ which takes into account the boundary conditions appearing in the definition of the space $\vec{V}$, so that the inclusion

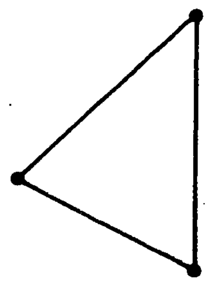
(2.1-3) $$\vec{V}_h \subset \vec{V}$$

holds.

More precisely, the different spaces are defined as follows :

(2.1-4)

<u>Space</u> X_{h1} is the finite element space associated to triangles



of type (1), i.e., the functions of X_{h1} are such that :

(i) on every $K \in \mathcal{T}_h$, they belong to $P_1(K)$;

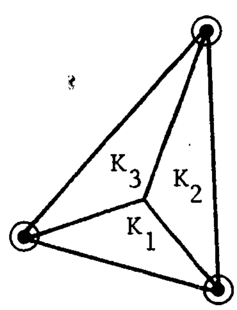
(ii) on every $K \in \mathcal{T}_h$, they are determined by their values at the vertices of K ;

(iii) $X_{h1} \subset \mathcal{C}^0(\bar{\Omega})$.

<u>Space</u> $V_{h1} = \{v_h \in X_{h1} \; ; \; v_h = 0 \text{ on } \Gamma_0\}$.

(2.1-5)

<u>Space</u> X_{h2} is the finite element space associated to reduced HSIEH-CLOUGH-TOCHER triangles (see CLOUGH-TOCHER [1]), i.e., the functions



of X_{h2} are such that :

(i) on every $K \in \mathcal{T}_h$, they belong to P_K with
$P_K = \{p \in \mathcal{C}^1(K) ; p|_{K_i} \in P_3(K_i),\ 1 \le i \le 3 ,$
$\partial_\nu p|_{K'} \in P_1(K')$ for each side K' of K$\}$

(ii) on every $K \in \mathcal{T}_h$, they are determined by their values and the values of their first partial derivatives at the vertices of K ;

(iii) $X_{h2} \subset \mathcal{C}^1(\bar{\Omega})$.

<u>Space</u> $V_{h2} = \{v_h \in X_{h2} \; ; \; v_h = \partial_n v_h = 0 \text{ on } \Gamma_0\}$.

In the previous definitions, we use the following notation : P_k is the space of all polynomials in ξ^1 and ξ^2 of degree $\leq k$. On the figures, the knowledge of the value of a function (resp. of its first derivatives) at a point is indicated by a black point (resp. a circle surrounding this point).

Finally, in the definitions of the spaces V_{h1} and V_{h2} we have assumed that the triangulations $\mathcal{T}_h$ are such that Γ_o is an exact union of sides of triangles, so that the inclusions

$$V_{h1} \subset V_1 = \{v \in H^1(\Omega) \ , \ v|_{\Gamma_o} = 0\}$$

and

$$V_{h2} \subset V_2 = \{v \in H^2(\Omega) \ , \ v|_{\Gamma_o} = \partial_n \, v|_{\Gamma_o} = 0\}$$

can be easily satisfied.

2.2 - The discrete problem

The corresponding discrete problem can be written as follows :

$$(2.2\text{-}1) \quad \begin{cases} \underline{\text{Find}}\ \vec{u}_h \in \vec{V}_h\ \underline{\text{such that}} \\ a(\vec{u}_h,\vec{v}_h) = f(\vec{v}_h) \quad , \quad \forall \vec{v}_h \in \vec{V}_h \end{cases}$$

The inclusion $\vec{V}_h \subset \vec{V}$ involves the theorem :

Theorem 2.2-1 : The problem (2.2-1) has one and only one solution.

□

Using CIARLET [2], we find the following error estimate :

Theorem 2.2-2 : If the solution $\vec{u}$ of the problem (1.2-5) belongs to the space $\vec{V} \cap [(H^2(\Omega))^2 \times H^3(\Omega)]$, then, there exists a constant C, independent of h, such that

$$(2.2\text{-}2) \qquad \|\vec{u}-\vec{u}_h\| \leq Ch \ \{|u_1|^2_{2,\Omega} + |u_2|^2_{2,\Omega} + |u_3|^2_{3,\Omega}\}^{1/2} \ .$$

□

3 - THE DISCRETE PROBLEM USING FLAT PLATE ELEMENTS AND COMPATIBILITY RELATIONS

In this paragraph, we analyze a nonconforming approximation of the geometry and so doing, we generalize the studies of CIARLET [1] and C. JOHNSON [1] relative to circular arches, and the study of BERNADOU-DUCATEL [1] relative to general arches, to the case of a portion of *right circular cylinder*.

First, we define the approximate middle surface $\bar{\mathcal{S}}_h$. Next, we introduce the discrete space $\vec{\vec{V}}_h$ using the compatibility conditions and finally, we set the discrete problem.

3.1 - The approximate middle surface $\bar{\mathcal{S}}_h$

In this section, we consider an approximation of the mapping $\vec{\phi} = \phi^i \vec{e}_i$ defined as follows : each component ϕ^i, $i = 1,2,3$, is replaced by its interpolant ϕ_h^i in the finite element space Φ_h such that

$$(3.1\text{-}1) \qquad \Phi_h \equiv X_{h1} \quad , \quad X_{h1} \text{ given by } (2.1\text{-}4) \; .$$

Let $\vec{\phi}_h \in (\Phi_h)^3$ be the interpolant of $\vec{\phi}$. This approximation amounts to replace the given middle surface $\mathcal{S}$ by a *faceted middle surface* $\bar{\mathcal{S}}_h$. By construction, the images of the vertices of the triangulation $\mathcal{C}_h$ of the reference set Ω by the mapping $\vec{\phi}_h$ are in the initial middle surface $\mathcal{S}$. These considerations are illustrated by Figure 3.1-1.

To each flat triangle $k = \vec{\phi}_h(K)$, $K \in \mathcal{C}_h$, we are able to associate, by analogy with section 1.1, local basis, fundamental forms, Christoffel symbols, i.e., with obvious notations (no summation on h) :

$$(3.1\text{-}2) \qquad \vec{a}_{h\alpha} = \vec{\phi}_{h,\alpha} \quad , \quad \vec{a}_{h3} = \frac{\vec{a}_{h1} \times \vec{a}_{h2}}{|\vec{a}_{h1} \times \vec{a}_{h2}|}$$

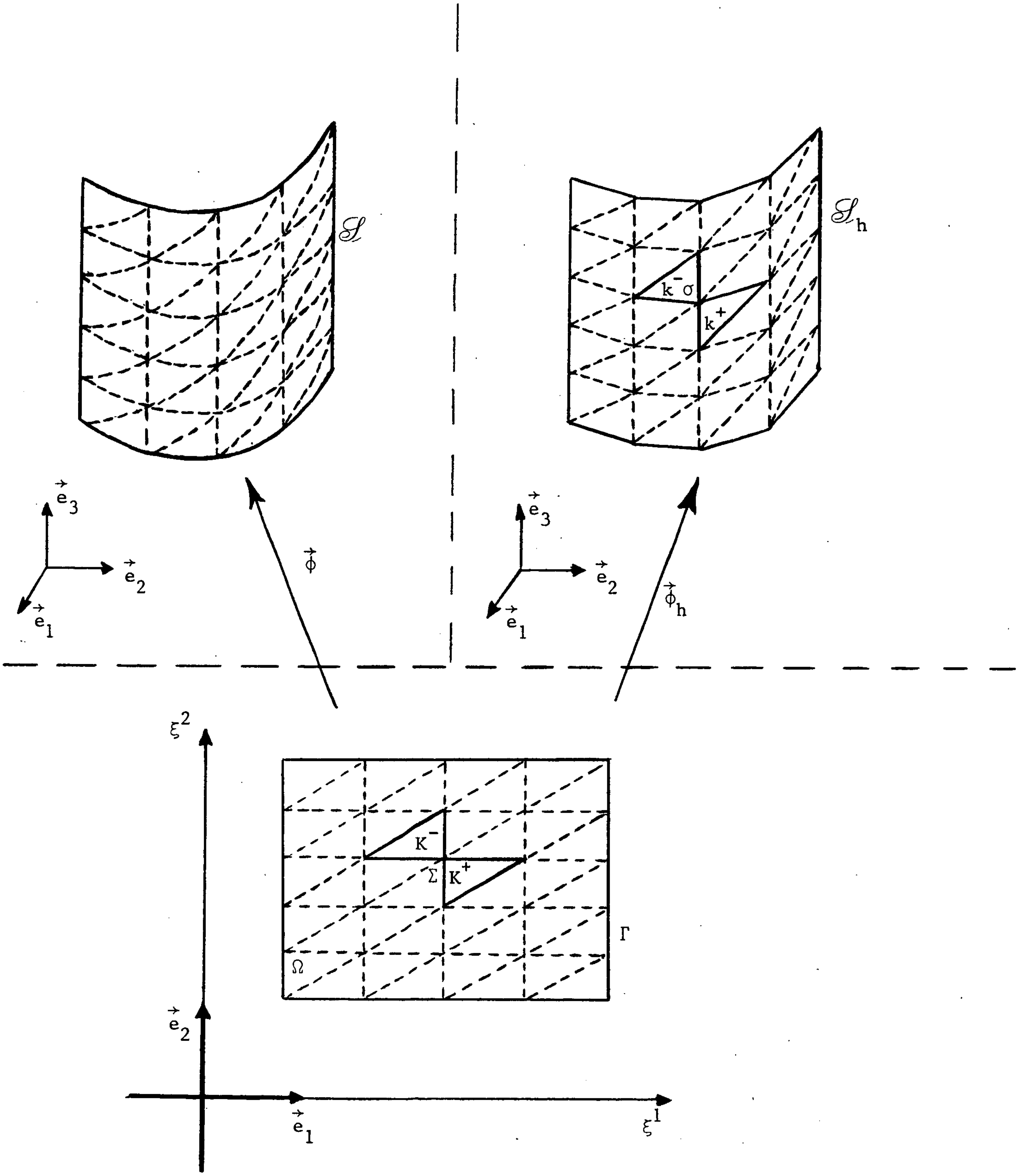


Figure 3.1-1 : The initial middle surface $\mathcal{S}$ and the faceted approximate middle surface $\mathcal{S}_h$.

(3.1-3) $\sqrt{a_h} = |\vec{a}_{h1} \times \vec{a}_{h2}|$

(3.1-4) $a_{h\alpha\beta} = \vec{a}_{h\alpha} \cdot \vec{a}_{h\beta}$,

(3.1-5) $\vec{a}_h^{\alpha} = a_h^{\alpha\beta} \vec{a}_{h\beta}$, $\vec{a}_h^3 = \vec{a}_{h3}$,

where the matrix $(a_h^{\alpha\beta})$ is the inverse of the matrix $(a_{h\alpha\beta})$,

(3.1-6) $b_{h\alpha\beta} = 0$, $b_{h\alpha}^{\beta} = 0$, $\Gamma_{h\alpha\beta}^{\lambda} = 0$.

It is worth to note that all the previous quantities are constant in any given triangle $K \in \mathcal{T}_h$ with possible discontinuities on the interfaces due to the discontinuity of the first derivatives of $\vec{\phi}_h$ on these interfaces.

3.2 - The discrete space $\vec{\tilde{V}}_h$ using compatibility conditions

One of the main ideas of the approximation of shell problems by flat plate elements is to approach the energy of the shell by a *sum of plate energies defined on each facet* of the approximate middle surface $\mathcal{S}_h$.

In this goal, we first introduce a discrete space

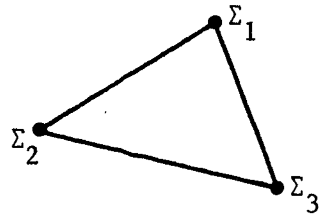
(3.2-1) $\vec{\tilde{X}}_h = \tilde{X}_{h1} \times \tilde{X}_{h1} \times \tilde{X}_{h2}$,

where the space $\tilde{X}_{h1}$ (resp. $\tilde{X}_{h2}$) will be used to approximate the components $\tilde{v}_{h1}$ and $\tilde{v}_{h2}$ (resp. the component $\tilde{v}_{h3}$) on the local basis $\vec{a}_{hi}$ of a displacement field $\vec{\tilde{v}}_h$.

Since the local basis are constant in any triangle $K \in \mathcal{T}_h$ with possible discontinuities on the interfaces, the *functions of the spaces* $\tilde{X}_{h\alpha}$ *will be determined triangle by triangle* without imposing any conditions of connections between functions defined on adjacent triangles.

More precisely, the spaces $\overset{\approx}{X}_{h1}$ and $\overset{\approx}{X}_{h2}$ are defined as follows :

(3.2-2) <u>Space</u> $\overset{\approx}{X}_{h1}$: the functions of the space $\overset{\approx}{X}_{h1}$ are independently defined triangle by triangle. On a given triangle $K \in \mathcal{T}_h$ with vertices Σ_1, Σ_2, Σ_3, the restriction $v_{h|K}$ of any function $v_h \in \overset{\approx}{X}_{h1}$ is such that

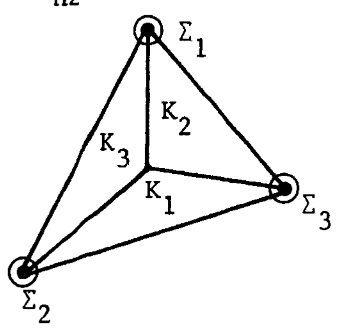


(i) $v_{h|K} \in P_1(K)$;

(ii) $v_{h|K}$ is completely specified by the values $v_{h|K}(\Sigma_i)$, $i = 1, 2, 3$.

The definition of the space X_{h2} uses the reduced HSIEH-CLOUGH-TOCHER triangle :

(3.2-3) <u>Space</u> $\overset{\approx}{X}_{h2}$: The functions of the space $\overset{\approx}{X}_{h2}$ are independently defined triangle by triangle. On a given triangle $K \in \mathcal{T}_h$ partitioned using three nondegenerated triangles K_i, the restriction $v_{h|K}$ of any function $v_h \in \overset{\approx}{X}_{h2}$ is such that



(i) $v_{h|K} \in P_K$ with

$P_K = \{p \in \mathcal{C}^1(K); p_{|K_i} \in P_3(K_i), 1 \le i \le 3, \partial_n p_{|K'} \in P_1(K')$ for each side K' of $K\}$;

(ii) $v_{h|K}$ is completely determined by the values $v_{h|K}(\Sigma_i)$ and $Dv_{h|K}(\Sigma_i)$, where Σ_i, $1 \le i \le 3$, denote the vertices of K.

These definitions of the spaces $\overset{\approx}{X}_{h1}$ and $\overset{\approx}{X}_{h2}$ involve immediately that the space $\overset{\approx}{\vec{X}}_h$ has $15M_h$ degrees of freedom, where M_h denotes the number of triangles of the triangulation.

<u>Derivation of compatibility conditions</u> :

To get an approximate energy which is consistent with the energy of the shell, we must introduce *constraints*, i.e., *compatibility conditions*, on the functions of the space $\overset{\approx}{\vec{X}}_h$.

To derive these compatibility conditions, it is convenient to consider two facets k^+ and k^- of the approximate middle surface $\mathcal{S}_h$, which have a common vertex σ (see Figure 3.1-1). This common vertex can be regarded as

(i) a point of the middle surface $\mathcal{S}$: then, using the notations of section 1.1, any displacement field $\vec{v} \in \vec{V}$ of the middle surface $\mathcal{S}$ has the following components at point σ, or, more conveniently, at point Σ (by using the mapping $\vec{\phi}$) :

$$(3.2\text{-}4) \qquad \vec{v}(\Sigma) = v_i(\Sigma)\, \vec{a}^i(\Sigma) \ .$$

To the displacement field $\vec{v}$, we associate the usual rotation vector $\vec{\omega}$

$$(3.2\text{-}5) \quad \left\{ \begin{array}{l} \vec{\omega}(\vec{v}) = \omega^i(\vec{v})\, \vec{a}_i \ , \\ \text{with } \omega^\lambda(\vec{v}) = \varepsilon^{\lambda\beta}(v_{3|\beta} + b^\alpha_\beta v_\alpha) \ , \ \omega^3(\vec{v}) = \frac{1}{2}\,\varepsilon^{\lambda\beta}\, v_{\beta|\lambda} \ , \\ \text{where } \varepsilon^{\alpha\beta} = \frac{1}{\sqrt{a}}\, e^{\alpha\beta} \ , \ (e^{\alpha\beta}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = (e_{\alpha\beta}) \ , \\ \text{i.e., by using (1.1-8) and (1.1-14)} \ , \\ \omega^1(\vec{v}) = \frac{1}{\sqrt{a}}\, v_{3,2} \ , \ \omega^2(\vec{v}) = -\frac{1}{\sqrt{a}}\,(v_{3,1} - \frac{1}{R}\, v_1) \ , \\ \qquad\qquad \omega^3(\vec{v}) = \frac{1}{2\sqrt{a}}\,(v_{2,1} - v_{1,2}) \end{array} \right.$$

To the displacement field $\vec{v}$, we also associate the strain tensor $(\gamma_{\alpha\beta})$ and the change of curvature tensor $(\rho_{\alpha\beta})$ which expressions are given by (1.1-15) and (1.1-16) respectively.

(ii) a vertex of the facet $k^+ = \vec{\phi}_h(K^+)$. For clarity, we shall use the superscript + to refer to parameters attached to the triangle K^+. Thus, we shall write Σ^+ instead of Σ, $\vec{a}^{i+}_h$ instead of $\vec{a}^i_h$, etc... Then, any displacement field $\vec{\tilde{\tilde{v}}}_h \in \vec{\tilde{\tilde{V}}}_h$ of the faceted surface $\mathcal{S}_h$ has the following components at point $\sigma^+ = \sigma$, or, using $\vec{\phi}_h$, at point Σ^+ :

$$(3.2\text{-}6) \qquad \vec{\tilde{\tilde{v}}}_h(\Sigma^+) = \tilde{\tilde{v}}_{hi}(\Sigma^+)\, \vec{a}^{i+}_h \ .$$

To the displacement field $\vec{\tilde{\tilde{v}}}_h$, we associate the usual rotation vector $\vec{\tilde{\tilde{\omega}}}_h$, the strain tensor $\tilde{\tilde{\gamma}}_{h\alpha\beta}$ and the change of curvature tensor $\tilde{\tilde{\rho}}_{h\alpha\beta}$ which expressions are obtained triangle by triangle using the mapping $\vec{\phi}_h$. By using relation (3.1-6) and by denoting $\vec{\tilde{\tilde{v}}}{}^+_h = \vec{\tilde{\tilde{v}}}_{h|K^+}$, we obtain for the triangle K^+ :

$$
(3.2\text{-}7)\quad \left\{ \begin{array}{l} \vec{\tilde{\tilde{\omega}}}_h(\vec{\tilde{\tilde{v}}}{}^+_h) = \tilde{\tilde{\omega}}{}^{i+}_h \, \vec{a}{}^+_{hi} \;, \\ \text{with } \tilde{\tilde{\omega}}{}^{\lambda+}_h = \dfrac{1}{\sqrt{a^+_h}}\, e^{\lambda\beta}\, \tilde{\tilde{v}}{}^+_{h3,\beta} \;, \; \tilde{\tilde{\omega}}{}^{3+}_h = \dfrac{1}{2\sqrt{a^+_h}}\, e^{\lambda\beta}\, \tilde{\tilde{v}}{}^+_{h\beta,\lambda} \end{array} \right.
$$

$$
(3.2\text{-}8)\qquad \tilde{\tilde{\gamma}}_{h\alpha\beta}(\vec{\tilde{\tilde{v}}}{}^+_h) = \frac{1}{2}\,(\tilde{\tilde{v}}{}^+_{h\beta,\alpha} + \tilde{\tilde{v}}{}^+_{h\alpha,\beta}) \;, \; \tilde{\tilde{\gamma}}{}^{\;\alpha}_{h\beta}(\vec{\tilde{\tilde{v}}}{}^+_h) = a^{\alpha\lambda+}_h \, \tilde{\tilde{\gamma}}_{h\lambda\beta}(\vec{\tilde{\tilde{v}}}{}^+_h) \;,
$$

$$
(3.2\text{-}9)\qquad \tilde{\tilde{\rho}}_{h\alpha\beta}(\vec{\tilde{\tilde{v}}}{}^+_h) = \tilde{\tilde{v}}{}^+_{h3,\alpha\beta} \;, \; \tilde{\tilde{\rho}}{}^{\;\alpha}_{h\beta}(\vec{\tilde{\tilde{v}}}{}^+_h) = a^{\alpha\lambda+}_h \, \tilde{\tilde{\rho}}_{h\lambda\beta}(\vec{\tilde{\tilde{v}}}{}^+_h) \;;
$$

(iii) a vertex of the facet $k^- = \vec{\phi}_h(K^-)$. We derive similar results replacing the superscript + by the superscript − .

Now, we are able to get the compatibility conditions which correspond to relations that the degrees of freedom of the space $\vec{\tilde{\tilde{X}}}_h$ have to satisfy. We derive them by writting that :

(i) <u>the displacement</u> $\vec{\tilde{\tilde{v}}}_h$ <u>is continuous at the vertices</u> σ <u>of the surface</u> $\mathcal{S}$, <u>or equivalently, at the vertices</u> Σ <u>of the triangulation</u> $\mathcal{T}_h$, i.e.,

$$
(3.2\text{-}10)\qquad \vec{\tilde{\tilde{v}}}_h(\Sigma^+) = \vec{\tilde{\tilde{v}}}_h(\Sigma^-) \quad , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h.
$$

Next, we remark that the knowledge of the <u>values of the degrees of freedom</u> of the function $\vec{\tilde{\tilde{v}}}_h \in \vec{\tilde{\tilde{X}}}_h$ involves the knowledge of the values of the rotation vectors $\vec{\tilde{\tilde{\omega}}}_h(\Sigma^+)$ and $\vec{\tilde{\tilde{\omega}}}_h(\Sigma^-)$, for any vertex Σ of $\mathcal{T}_h$. Indeed, using relation (3.2-7), the result is immediate for the components $\tilde{\tilde{\omega}}{}^{\lambda+}_h$. For the third component, we have just to remember that $\tilde{\tilde{v}}{}^+_{h\beta} \in P_1(K^+)$ and hence the values $\tilde{\tilde{v}}{}^+_{h\beta,\lambda}(\Sigma^+)$ only depend on the values of $\tilde{\tilde{v}}{}^+_{h\beta}$ at the three vertices of the triangle K^+. Obviously, this argument can be applied in the same way to prove that $\vec{\tilde{\tilde{\omega}}}_h(\Sigma^-)$ is well known. Hence we can state the second kind of conditions :

(ii) the tangential components - with respect to the middle surface $\mathcal{S}$ - of the rotation vector $\vec{\tilde{\tilde{\omega}}}_h(\vec{\tilde{\tilde{v}}}_h)$ are continuous at the vertices σ of the surface $\mathcal{S}$ (or $\mathcal{S}_h$), or equivalently, at the vertices Σ of $\mathcal{T}_h$, i.e.,

$$\text{(3.2-11)} \qquad \vec{\tilde{\tilde{\omega}}}_h(\Sigma^+).\vec{a}^{\alpha}(\Sigma) = \vec{\tilde{\tilde{\omega}}}_h(\Sigma^-).\vec{a}^{\alpha}(\Sigma) \quad , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h \, .$$

These conditions (3.2-10) (3.2-11) seem to be natural enough from mechanical viewpoint. They correspond to the conditions used by CLOUGH-JOHNSON [1,2] and we shall see that they are sufficient to insure the convergence of this flat element method.

Let $\vec{\tilde{X}}_h$ be the following space

$$\text{(3.2-12)} \qquad \vec{\tilde{X}}_h = \{\vec{\tilde{\tilde{v}}}_h \in \vec{\tilde{\tilde{X}}}_h \; ; \; \vec{\tilde{\tilde{v}}}_h \text{ satisfies the compatibility conditions (3.2-10) (3.2-11)}\}$$

and let us state the following hypothesis :

Hypothesis 3.2-1 : Let $\sigma = \sigma^+ = \sigma^-$ be a vertex of the faceted surface $\mathcal{S}_h$ respectively considered as a point of $\mathcal{S}$, as a vertex of a facet k^+ and as a vertex of a facet k^-. Denoting $\vec{a}^3(\Sigma)$, $\vec{a}_h^{3+}$ and $\vec{a}_h^{3-}$ the corresponding normal vector to the middle surface $\mathcal{S}$, to the facet k^+ and to the facet k^-, then we assume

$$\text{(3.2-13)} \qquad \vec{a}^3(\Sigma).\vec{a}_h^{3+} \neq 0 \quad , \quad \vec{a}^3(\Sigma).\vec{a}_h^{3-} \neq 0 \; , \; \forall \Sigma \text{ vertex of two adjacent triangles } K^+ \text{ and } K^- .$$

□

Remark 3.2-1 : If the regular families of triangulations $\mathcal{T}_h$ are chosen so that any triangle of $\mathcal{T}_h$ has one side parallel to the cylinder axis (we shall prove the convergence in this case), then Hypothesis 3.2-1 is satisfied provided that the length of the projection of the triangle on ξ^1-axis is strictly less than π.

Indeed, let us consider a triangle K^+ with vertex $\Sigma_i^+ = (\xi_i^{1+}, \xi_i^{2+})$, $1 \le i \le 3$, such that $\xi_1^{1+} = \xi_3^{1+}$ (see Figure 3.2-1).

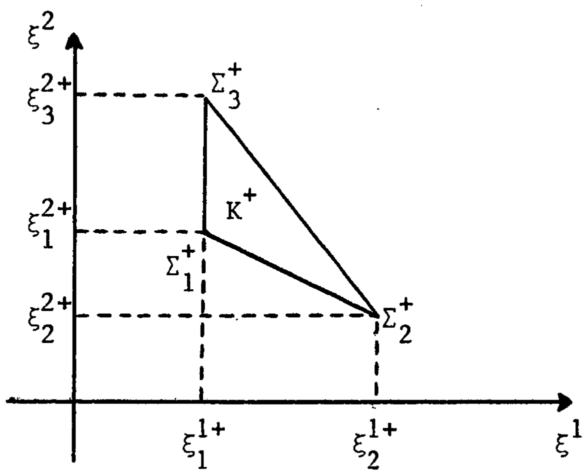


Figure 3.2-1

For such a triangle, we have

$$
(3.2\text{-}14)\quad \begin{cases}
\lambda_1 = \dfrac{1}{(\xi_2^{1+}-\xi_1^{1+})(\xi_3^{2+}-\xi_1^{2+})} \{(\xi_3^{2+}-\xi_2^{2+})(\xi_2^{1+}-\xi^1) + (\xi_2^{1+}-\xi_1^{1+})(\xi_2^{2+}-\xi^2)\} \\[2ex]
\lambda_2 = \dfrac{1}{(\xi_2^{1+}-\xi_1^{1+})(\xi_3^{2+}-\xi_1^{2+})} (\xi_3^{2+}-\xi_1^{2+})(\xi^1-\xi_1^{1+}) \\[2ex]
\lambda_3 = \dfrac{1}{(\xi_2^{1+}-\xi_1^{1+})(\xi_3^{2+}-\xi_1^{2+})} \{(\xi_1^{2+}-\xi_2^{2+})(\xi^1-\xi_1^{1+}) + (\xi_2^{1+}-\xi_1^{1+})(\xi^2-\xi_1^{2+})\}
\end{cases}
$$

and

$$
\vec{\phi}_h\Big|_{K^+} = \lambda_1\, \vec{\phi}(\Sigma_1^+) + \lambda_2\, \vec{\phi}(\Sigma_2^+) + \lambda_3\, \vec{\phi}(\Sigma_3^+) \ .
$$

From (3.1-2), we obtain on triangle K^+ :

$$\vec{a}^{+}_{h1} = \frac{2R \sin \frac{\xi_2^{1+}-\xi_1^{1+}}{2}}{\xi_2^{1+}-\xi_1^{1+}} \left\{ \begin{array}{l} -\sin \frac{\xi_1^{1+}+\xi_2^{1+}}{2} \\ \cos \frac{\xi_1^{1+}+\xi_2^{1+}}{2} \\ 0 \end{array} \right.$$

$$\vec{a}^{+}_{h2} = \left\{ \begin{array}{l} 0 \\ 0 \\ 1 \end{array} \right.$$

$$a^{+}_{h} = \frac{4R^2}{(\xi_2^{1+}-\xi_1^{1+})^2} \sin^2 \frac{\xi_2^{1+}-\xi_1^{1+}}{2}$$

so that

$$\vec{a}^{+}_{h3} = \left\{ \begin{array}{l} \cos \frac{\xi_1^{1+}+\xi_2^{1+}}{2} \\ \sin \frac{\xi_1^{1+}+\xi_2^{1+}}{2} \\ 0 \end{array} \right.$$

Then, from (1.1-2), we derive

$$\vec{a}^3(\Sigma_i^+).\vec{a}^{+}_{h3} = \cos \frac{\xi_2^{1+}-\xi_1^{1+}}{2} > 0 \quad , \; i = 1,2,3 \; ,$$

as soon as $0 < \xi_2^{1+}-\xi_1^{1+} < \pi$.

□

Now we are going to prove the following theorem which will be very useful for the next purposes :

Theorem 3.2-1 : Under the Hypothesis 3.2-1, there exists a bijection F_h between the spaces $\vec{\vec{X}}_h$ and $\vec{X}_h$.

<u>Proof</u> : (i) $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h \longrightarrow \vec{v}_h \in \vec{X}_h$:

Let us consider any function $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$. Then, to the set of values of the degrees of freedom of the function $\vec{\tilde{v}}_h$, we associate the following set of values :

$$\begin{cases} \vec{A}(\Sigma) = \vec{\tilde{v}}_h(\Sigma^+) = \vec{\tilde{v}}_h(\Sigma^-) \quad , \forall\Sigma \text{ vertex of } \mathcal{T}_h , \\ B^\alpha(\Sigma) = \vec{\tilde{\omega}}_h(\Sigma^+).\vec{a}^\alpha(\Sigma) = \vec{\tilde{\omega}}_h(\Sigma^-).\vec{a}^\alpha(\Sigma) \quad , \forall\Sigma \text{ vertex of } \mathcal{T}_h . \end{cases}$$

To this set of values, we associate the function $\vec{v}_h \in \vec{X}_h$ determined by the following values of its degrees of freedom :

$$\begin{cases} \vec{v}_h(\Sigma) = \vec{A}(\Sigma) \quad , \forall\Sigma \text{ vertex of } \mathcal{T}_h , & (3.2\text{-}15) \\ v_{h3,\nu}(\Sigma) = \varepsilon_{\alpha\nu}(\Sigma)\, B^\alpha(\Sigma) - b^\alpha_\nu(\Sigma)\, \vec{A}(\Sigma)\, \vec{a}_\alpha(\Sigma) \quad , \forall\Sigma \text{ vertex of } \mathcal{T}_h , & (3.2\text{-}16) \end{cases}$$

where $\varepsilon_{\alpha\nu} = \sqrt{a}\, e_{\alpha\nu}$, $e_{\alpha\nu}$ being defined in (3.2-5).

Thus, to any given function $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$, we can associate through the relations (3.2-15) and (3.2-16) one and only one function $\vec{v}_h \in \vec{X}_h$.

Now, let us consider the converse property :

(ii) $\vec{v}_h \in \vec{X}_h \longrightarrow \vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$:

The knowledge of $\vec{v}_h \in \vec{X}_h$ is equivalent to the knowledge of its degrees of freedom $\vec{v}_h(\Sigma)$ and $v_{h3,\nu}(\Sigma)$ for all vertices Σ of $\mathcal{T}_h$. To this set of values, we associate the following set of values (compare with (3.2-5)) :

$$\begin{cases} \vec{A}(\Sigma) = \vec{v}_h(\Sigma) \quad , \forall\Sigma \text{ vertex of } \mathcal{T}_h , & (3.2\text{-}17) \\ B^\lambda(\Sigma) = \varepsilon^{\lambda\beta}[v_{h3,\beta}(\Sigma) + b^\alpha_\beta(\Sigma)\, v_{h\alpha}(\Sigma)] \quad , \forall\Sigma \text{ vertex of } \mathcal{T}_h & (3.2\text{-}18) \end{cases}$$

Then, we take the following values for the first part of the set of the degrees of freedom of the searched function $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$:

$$(3.2\text{-}19) \qquad \vec{\tilde{v}}_h(\Sigma^+) = \vec{\tilde{v}}_h(\Sigma^-) = \vec{A}(\Sigma) \ .$$

Applying (3.2-7), we can derive from these values the values of

$$(3.2\text{-}20) \qquad \tilde{\omega}_h^3(\Sigma^+) = \frac{1}{2\sqrt{a_h^+}}\, e^{\lambda\beta}\, \tilde{v}^+_{h\beta,\lambda}(\Sigma^+) \quad , \quad \tilde{\omega}_h^3(\Sigma^-) = \frac{1}{2\sqrt{a_h^-}}\, e^{\lambda\beta}\, \tilde{v}^-_{h\beta,\lambda}(\Sigma^-) \ ,$$

using the analysis which follows the relations (3.2-10).

Then, we are able to determine the components $\tilde{\omega}_h^\beta(\Sigma^+)$ and $\tilde{\omega}_h^\beta(\Sigma^-)$ by writting that the vectors $\vec{\tilde{\omega}}_h(\Sigma^+) = \tilde{\omega}_h^i(\Sigma^+)\vec{a}^+_{hi}$ and $\vec{\tilde{\omega}}_h(\Sigma^-) = \tilde{\omega}_h^i(\Sigma^-)\vec{a}^-_{hi}$ are the unique solutions of the system

$$(3.2\text{-}21) \qquad \vec{\tilde{\omega}}_h(\Sigma^+).\vec{a}^\lambda(\Sigma) = \vec{\tilde{\omega}}_h(\Sigma^-).\vec{a}^\lambda(\Sigma) = B^\lambda(\Sigma) \quad , \forall\Sigma \ \text{ vertex of } \mathscr{T}_h .$$

Leaving aside the second term of above equation, we find

$$\tilde{\omega}_h^\beta(\Sigma^+)\vec{a}^+_{h\beta}.\vec{a}^\lambda(\Sigma) = B^\lambda(\Sigma) - \tilde{\omega}_h^3(\Sigma^+)\vec{a}^+_{h3}.\vec{a}^\lambda(\Sigma) \ .$$

These equations permit to derive $\tilde{\omega}_h^\beta(\Sigma^+)$ if we observe that $\det(\vec{a}^+_{h\beta}.\vec{a}^\lambda(\Sigma)) \neq 0$. Indeed, we notice that $\vec{a}^\lambda(\Sigma) = (\vec{a}^+_{hi}.\vec{a}^\lambda(\Sigma))\vec{a}_h^{i+}$ and thus, on the one hand,

$$(\vec{a}^1(\Sigma) \times \vec{a}^2(\Sigma))\cdot\vec{a}_h^{3+} = \frac{1}{\sqrt{a(\Sigma)}}\,(\vec{a}^3(\Sigma).\vec{a}_h^{3+})$$

and, on the other hand,

$$(\vec{a}^1(\Sigma) \times \vec{a}^2(\Sigma))\cdot\vec{a}_h^{3+} = \frac{1}{\sqrt{a_h^+}}\det(\vec{a}^+_{h\beta}.\vec{a}^\lambda(\Sigma)).$$

Then, the result arises from the relation (3.2-13) of the Hypothesis 3.2-1. Exactly in the same way, we can derive the components $\tilde{\omega}_h^\beta(\Sigma^-)$.

Finally, from $\tilde{\omega}_h^\beta(\Sigma^+)$, $\tilde{\omega}_h^\beta(\Sigma^-)$ and relations (3.2-7), we get

$$(3.2\text{-}22) \qquad \tilde{v}_{h3,\beta}(\Sigma^+) = e_{\lambda\beta}\sqrt{a_h^+}\,\tilde{\omega}_h^\lambda(\Sigma^+) \quad , \quad \tilde{v}_{h3,\beta}(\Sigma^-) = e_{\lambda\beta}\sqrt{a_h^-}\,\tilde{\omega}_h^\lambda(\Sigma^-) .$$

Then, the relations (3.2-19)(3.2-22) determine one and only one function $\vec{\tilde{v}}_h \in \vec{\tilde{\tilde{X}}}_h$. Since by definition, this function satisfies the relations (3.2-19)(3.2-21), i.e., the compatibility relations, the function $\vec{\tilde{v}}_h$ belongs in fact to the subspace $\vec{\tilde{X}}_h$.

□

Remark 3.2-2 : In general, we do not have $\vec{\tilde{X}}_h \subset (H^1(\Omega))^2 \times H^2(\Omega)$ but only

$$\vec{\tilde{X}}_h \subset \prod_{K \in \mathcal{T}_h} (H^1(K) \times H^1(K) \times H^2(K)) .$$

Naturally, by construction, for any $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$, the corresponding function $\vec{v}_h$ through the bijection defined in theorem 3.2-1 is such that

$$\vec{v}_h \in \mathcal{C}^0(\bar{\Omega}) \times \mathcal{C}^0(\bar{\Omega}) \times \mathcal{C}^1(\bar{\Omega}) .$$

□

Definition of the space $\vec{\tilde{V}}_h$:

To study the convergence of the flat plate element method, it is useful to define a subspace $\vec{\tilde{V}}_h$ of the space $\vec{\tilde{X}}_h$ which image, through the bijection F_h, is precisely the space $\vec{V}_h$ defined in (2.1-3)(2.1-4)(2.1-5), i.e.,

$$(3.2\text{-}23) \qquad \vec{V}_h = \{\vec{v}_h \in \vec{X}_h \quad ; \quad \vec{v}_{h|\Gamma_o} = \vec{0} \quad , \quad \partial_n v_{h3} = 0 \text{ on } \Gamma_o\} .$$

According to the definition of the space $\vec{X}_h$, to get the boundary conditions $\vec{v}_{h|\Gamma_o} = \vec{0}$ and $\partial_n v_{h3} = 0$ on Γ_o, it suffices to equal to zero the following degrees of freedom :

$$(3.2\text{-}24)\quad \begin{cases} \vec{v}_h(\Sigma) = \vec{0} \\ v_{h3,\nu}(\Sigma) = 0 \end{cases} \qquad \forall\Sigma \text{ vertex of } \mathcal{C}_h \text{ located on } \Gamma_o \ .$$

Using the relations (3.2-17)(3.2-18), the equations (3.2-24) involve

$$\begin{aligned} &(3.2\text{-}25) \\ &(3.2\text{-}26) \end{aligned}\quad \begin{cases} \vec{A}(\Sigma) = \vec{0} \\ B^{\lambda}(\Sigma) = 0 \end{cases} \qquad \forall\Sigma \text{ vertex of } \mathcal{C}_h \text{ located on } \Gamma_o \ .$$

The relation (3.2-25) combined with (3.2-19) yields

$$(3.2\text{-}27)\qquad \vec{\tilde{v}}_h(\Sigma) = \vec{0} \quad , \quad \forall\Sigma \text{ vertex of } \mathcal{C}_h \text{ located on } \Gamma_o \ .$$

Particularly, this last relation and the definition of the space $\tilde{X}_{h1}$ involve

$$(3.2\text{-}28)\qquad \tilde{v}_{h\alpha} = 0 \quad \text{on } \Gamma_o \ , \ \alpha = 1,2 \ .$$

The relation (3.2-26) combined with (3.2-21) implies $\vec{\tilde{\omega}}_h(\Sigma^+).\vec{a}^{\lambda}(\Sigma) = 0$, and hence

$$\vec{\tilde{\omega}}_h(\Sigma^+) = C^+\vec{a}^3(\Sigma) \ ,$$

where C^+ denotes a real number. If we consider the components of the rotation vector $\vec{\tilde{\omega}}_h(\Sigma^+)$ on the basis $\vec{a}^+_{hi}$, then we obtain

$$\tilde{\omega}^i_h(\Sigma^+) = C^+\vec{a}^3(\Sigma).\vec{a}^{i+}_h \ ,$$

or by using (3.2-7)

$$(3.2\text{-}29)\qquad \frac{1}{\sqrt{a^+_h}}\, e^{\lambda\beta}\, \tilde{v}_{h3,\beta}(\Sigma^+) = C^+\vec{a}^3(\Sigma).\vec{a}^{\lambda+}_h \quad ,$$

$$(3.2\text{-}30) \qquad \frac{1}{2\sqrt{a_h^+}}\, e^{\lambda\beta}\, \tilde{v}_{h\beta,\lambda}(\Sigma^+) = C^+ \vec{a}^3(\Sigma).\vec{a}_h^{3+} \quad .$$

The Hypothesis 3.2-1 and the equation (3.2-30) determine the values of the constant C^+ on the triangle K^+ :

$$(3.2\text{-}31) \qquad C^+ = \frac{e^{\lambda\beta}}{2\sqrt{a_h^+}} \, \frac{\tilde{v}_{h\beta,\lambda}(\Sigma^+)}{\vec{a}^3(\Sigma).\vec{a}_h^{3+}} \quad ,$$

where, following a previous remark, $\tilde{v}_{h\beta,\lambda}(\Sigma^+)$ is only depending on the values of the functions $\tilde{v}_{h\beta}$ at the vertices of triangle K^+. Then, the relations (3.2-29)(3.2-31) determine $\tilde{v}_{h3,\beta}(\Sigma^+)$, i.e.,

$$(3.2\text{-}32) \qquad \tilde{v}_{h3,\beta}(\Sigma^+) = \frac{1}{2}\, e_{\lambda\beta}\, e^{\nu\mu} \, \frac{\vec{a}^3(\Sigma).\vec{a}_h^{\lambda+}}{\vec{a}^3(\Sigma).\vec{a}_h^{3+}} \, \tilde{v}_{h\mu,\nu}(\Sigma^+) \ , \ \forall\Sigma \text{ vertex of } \mathcal{C}_h \text{ located on } \Gamma_o \quad .$$

This last relation prove that in general $\tilde{v}_{h3,\beta}(\Sigma^+) \neq 0$ and hence

$$\tilde{v}_{h3|\Gamma_o} \neq 0 \quad \text{and} \quad \partial_n \tilde{v}_{h3|\Gamma_o} \neq 0 \quad .$$

Thus, the bijection $F_h^{-1} : \vec{X}_h \to \vec{\tilde{X}}_h$ associates to the subspace $\vec{V}_h$ of $\vec{X}_h$ the following subspace $\vec{\tilde{V}}_h$ of $\vec{\tilde{X}}_h$:

$$\vec{\tilde{V}}_h = \{\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h \ ; \ \vec{\tilde{v}}_h \text{ satisfies the relations (3.2-27)(3.2-32)}\} \ .$$

Particularly, this result means that the bijection F_h^{-1} associates to boundary conditions of <u>clamped type</u> on functions of the space $\vec{V}_h$, a set of boundary conditions which can be <u>different of clamped type</u> on functions of the space $\vec{\tilde{V}}_h$.

3.3 - The discrete problem

Now, we are able to define the discrete problem

$$(3.3\text{-}1)\quad \begin{cases} \text{Find } \vec{\vec{u}}_h \in \vec{\vec{V}}_h \text{ such that} \\ \tilde{a}_h(\vec{\vec{u}}_h,\vec{\vec{v}}_h) = f_h(\vec{v}_h) \;, \quad \forall \vec{\vec{v}}_h \in \vec{\vec{V}}_h \;, \end{cases}$$

with

$$(3.3\text{-}2)\quad \begin{cases} \tilde{a}_h(\vec{\vec{u}}_h,\vec{\vec{v}}_h) = \sum_{K\in\mathcal{T}_h} \int_K \frac{Ee}{1-\nu^2} \{(1-\nu)\tilde{\gamma}^{\ \alpha}_{h\beta}(\vec{\vec{u}}_h)\tilde{\gamma}^{\ \beta}_{h\alpha}(\vec{\vec{v}}_h) + \nu\tilde{\gamma}^{\ \alpha}_{h\alpha}(\vec{\vec{u}}_h)\tilde{\gamma}^{\ \beta}_{h\beta}(\vec{\vec{v}}_h) + \\ \qquad + \frac{e^2}{12}[(1-\nu)\tilde{\rho}^{\ \alpha}_{h\beta}(\vec{\vec{u}}_h)\tilde{\rho}^{\ \beta}_{h\alpha}(\vec{\vec{v}}_h) + \nu\tilde{\rho}^{\ \alpha}_{h\alpha}(\vec{\vec{u}}_h)\tilde{\rho}^{\ \beta}_{h\beta}(\vec{\vec{v}}_h)]\} \sqrt{a_h}\, d\xi^1 d\xi^2 \end{cases}$$

where the tensors $\tilde{\gamma}^{\ \alpha}_{h\beta}$ and $\tilde{\rho}^{\ \alpha}_{h\beta}$ are defined in (3.2-8)(3.2-9) and where the functions $\vec{v}_h \in \vec{V}_h$ and $\vec{\vec{v}}_h \in \vec{\vec{V}}_h$ are in correspondence through the bijection F_h defined in Theorem 3.2-1, and with

$$(3.3\text{-}3)\qquad f_h(\vec{v}_h) = \sum_{K\in\mathcal{T}_h} \frac{\text{meas}(K)}{3} \sum_{i=1}^{3} (\vec{p}.\vec{v}_h\ \sqrt{a})(\Sigma_i) \quad ,$$

where Σ_i, $i = 1,2,3$, denote the vertices of triangle K .

Using bijection F_h, the discrete problem (3.3-1) can be rewritten as a discrete problem defined on the space $\vec{V}_h$:

$$(3.3\text{-}4)\quad \begin{cases} \text{Find } \vec{u}_h \in \vec{V}_h \text{ such that} \\ a_h(\vec{u}_h,\vec{v}_h) = f_h(\vec{v}_h) \quad , \quad \forall \vec{v}_h \in \vec{V}_h \end{cases}$$

where, by definition, we set, for any $\vec{v}_h$, $\vec{w}_h \in \vec{V}_h$ (respectively in correspondence with $\vec{\vec{v}}_h$, $\vec{\vec{w}}_h \in \vec{\vec{V}}_h$ through the bijection F_h)

$$(3.3\text{-}5)\qquad a_h(\vec{v}_h,\vec{w}_h) = \tilde{a}_h(\vec{\vec{v}}_h,\vec{\vec{w}}_h) \quad .$$

4 - CONVERGENCE AND ERROR ESTIMATES

In this paragraph, we are going first to prove that the problem (3.3-4) has a unique solution $\vec{u}_h \in \vec{V}_h$ by showing the $\vec{V}_h$-ellipticity of the bilinear form $a_h(.,.)$. Then, we shall prove the existence of a constant C, independent of h, such that

$$\|\vec{u} - \vec{u}_h\|_{\vec{V}} \leq Ch$$

(i.e. the same order of convergence as in the conforming method defined in paragraph 2), where $\vec{u}$ (resp. $\vec{u}_h$) denotes the solution of the continuous problem (1.2-5) (resp. of the discrete problem (3.3-4)).

In order to obtain such an estimate, we first give an "abstract" error estimate.

4.1 - Abstract error estimate

Theorem 4.1-1 : Let us consider a family of discrete problems (3.3-4) for which the bilinear forms $a_h(.,.)$ are $\vec{V}_h$-elliptic, uniformly with respect to h, in the sense that there exists a constant $\alpha > 0$, independent of h, such that

$$\alpha\|\vec{v}_h\|^2_{\vec{V}} \leq a_h(\vec{v}_h,\vec{v}_h) \quad , \quad \forall \vec{v}_h \in \vec{V}_h \quad . \tag{4.1-1}$$

Then, there exists a constant C, independent of h, such that

$$\left\{\begin{aligned} \|\vec{u} - \vec{u}_h\|_{\vec{V}} \leq C \Big\{ \inf_{\vec{v}_h\in\vec{V}_h} \Big(\|\vec{u} - \vec{v}_h\|_{\vec{V}} + \sup_{\vec{w}_h\in\vec{V}_h} \frac{|a(\vec{v}_h,\vec{w}_h)-a_h(\vec{v}_h,\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{V}}} \Big) + \\ + \sup_{\vec{w}_h\in\vec{V}_h} \frac{|f(\vec{w}_h)-f_h(\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{V}}} \Big\} \end{aligned}\right. \tag{4.1-2}$$

where $\vec{u}$ (resp. $\vec{u}_h$) denotes the solution of the continuous problem (1.2-5) (resp. of the discrete problem (3.3-4)).

Proof : From the assumption of uniform $\vec{V}_h$-ellipticity, we derive the existence and the uniqueness of the solution $\vec{u}_h$ of the discrete problem (3.3-4). Then, let $\vec{v}_h$ be any element of the space $\vec{V}_h$. Since $\vec{V}_h \subset \vec{V}$, we can write

$$\alpha\|\vec{u}_h - \vec{v}_h\|^2_{\vec{V}} \leq a_h(\vec{u}_h-\vec{v}_h,\vec{u}_h-\vec{v}_h)$$

$$= a(\vec{u}-\vec{v}_h,\vec{u}_h-\vec{v}_h) + [a(\vec{v}_h,\vec{u}_h-\vec{v}_h) - a_h(\vec{v}_h,\vec{u}_h-\vec{v}_h)] +$$

$$+ [f_h(\vec{u}_h-\vec{v}_h) - f(\vec{u}_h-\vec{v}_h)] \quad ,$$

so that the continuity of the bilinear form a(.,.) involves

$$\alpha\|\vec{u}_h - \vec{v}_h\|_{\vec{V}} \leq M\|\vec{u} - \vec{v}_h\|_{\vec{V}} + \frac{|a(\vec{v}_h,\vec{u}_h-\vec{v}_h) - a_h(\vec{v}_h,\vec{u}_h-\vec{v}_h)|}{\|\vec{u}_h-\vec{v}_h\|_{\vec{V}}} +$$

$$+ \frac{|f(\vec{u}_h-\vec{v}_h) - f_h(\vec{u}_h-\vec{v}_h)|}{\|\vec{u}_h-\vec{v}_h\|_{\vec{V}}} \quad ,$$

and thus,

$$\alpha\|\vec{u}_h - \vec{v}_h\|_{\vec{V}} \leq M\|\vec{u} - \vec{v}_h\|_{\vec{V}} + \sup_{\vec{w}_h\in\vec{V}_h} \frac{|a(\vec{v}_h,\vec{w}_h) - a_h(\vec{v}_h,\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{V}}} +$$

$$+ \sup_{\vec{w}_h\in\vec{V}_h} \frac{|f(\vec{w}_h) - f_h(\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{V}}} \quad .$$

Combining this last inequality with the triangular inequality and taking the minimum with respect to $\vec{v}_h \in \vec{V}_h$, we get the inequality (4.1-2).

□

Let us remark that, in estimate (4.1-2), we find in addition to the usual approximation theory term $\inf_{\vec{v}_h\in\vec{V}_h} \|\vec{u} - \vec{v}_h\|_{\vec{V}}$, two additionnal terms which measure the consistency between the bilinear forms a(.,.) and $a_h(.,.)$ on the one hand, and between the linear forms f(.) and $f_h(.)$ on the other hand.

4.2 - Estimate of $|a(\vec{v}_h,\vec{w}_h) - a_h(\vec{v}_h,\vec{w}_h)|$

In order to estimate the consistency error $|a(\vec{v}_h,\vec{w}_h) - a_h(\vec{v}_h,\vec{w}_h)|$, let us consider any functions $\vec{v}_h,\vec{w}_h \in \vec{X}_h$ and let us denote $\tilde{\vec{v}}_h,\tilde{\vec{w}}_h$ respectively the corresponding functions in $\tilde{\vec{X}}_h$ through the bijection F_h defined in theorem 3.2-1. Using definition (3.3-5), we have

$$a(\vec{v}_h,\vec{w}_h) - a_h(\vec{v}_h,\vec{w}_h) = a(\vec{v}_h,\vec{w}_h) - \tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h) \tag{4.2-1}$$

with

$$\left\{\begin{aligned} a(\vec{v}_h,\vec{w}_h) = \sum_{K\in\mathcal{T}_h} \int_K \frac{Ee}{1-\nu^2}\, \{(1-\nu)\gamma^\alpha_\beta(\vec{v}_h)\gamma^\beta_\alpha(\vec{w}_h) + \nu\gamma^\alpha_\alpha(\vec{v}_h)\gamma^\beta_\beta(\vec{w}_h) + \\ + \frac{e^2}{12}\,[(1-\nu)\rho^\alpha_\beta(\vec{v}_h)\rho^\beta_\alpha(\vec{w}_h) + \nu\rho^\alpha_\alpha(\vec{v}_h)\rho^\beta_\beta(\vec{w}_h)]\}\sqrt{a}\, d\xi^1\, d\xi^2 \end{aligned}\right. \tag{4.2-2}$$

and

$$\left\{\begin{aligned} \tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h) = \sum_{K\in\mathcal{T}_h} \int_K \frac{Ee}{1-\nu^2}\, \{(1-\nu)\tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h) + \nu\tilde{\gamma}_{h\alpha}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\beta}^{\ \beta}(\tilde{\vec{w}}_h) + \\ + \frac{e^2}{12}\,[(1-\nu)\tilde{\rho}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\rho}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h) + \nu\tilde{\rho}_{h\alpha}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\rho}_{h\beta}^{\ \beta}(\tilde{\vec{w}}_h)]\}\sqrt{a_h}\, d\xi^1\, d\xi^2 \end{aligned}\right. , \tag{4.2-3}$$

so that we can write

$$|a(\vec{v}_h,\vec{w}_h) - \tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)| \le \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{4} |ER_{K_i}(\vec{v}_h,\vec{w}_h)| \tag{4.2-4}$$

with

$$ER_{K_1}(\vec{v}_h,\vec{w}_h) = \int_K \frac{Ee}{1+\nu}\, [\gamma^\alpha_\beta(\vec{v}_h)\gamma^\beta_\alpha(\vec{w}_h)\sqrt{a} - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)\sqrt{a_h}]\, d\xi^1 d\xi^2 \tag{4.2-5}$$

$$ER_{K_2}(\vec{v}_h,\vec{w}_h) = \int_K \frac{Ee\nu}{1-\nu^2}\, [\gamma^\alpha_\alpha(\vec{v}_h)\gamma^\beta_\beta(\vec{w}_h)\sqrt{a} - \tilde{\gamma}_{h\alpha}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\beta}^{\ \beta}(\tilde{\vec{w}}_h)\sqrt{a_h}]\, d\xi^1 d\xi^2, \tag{4.2-6}$$

$$ER_{K_3}(\vec{v}_h,\vec{w}_h) = \int_K \frac{Ee^3}{12(1+\nu)}\, [\rho^\alpha_\beta(\vec{v}_h)\rho^\beta_\alpha(\vec{w}_h)\sqrt{a} - \tilde{\rho}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\rho_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)\sqrt{a_h}]\, d\xi^1 d\xi^2, \tag{4.2-7}$$

$$ER_{K_4}(\vec{v}_h,\vec{w}_h) = \int_K \frac{Ee^3\nu}{12(1-\nu^2)}\, [\rho^\alpha_\alpha(\vec{v}_h)\rho^\beta_\beta(\vec{w}_h)\sqrt{a} - \tilde{\rho}_{h\alpha}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\rho}_{h\beta}^{\ \beta}(\tilde{\vec{w}}_h)\sqrt{a_h}]\, d\xi^1 d\xi^2. \tag{4.2-8}$$

In order to estimate the four previous terms, we are going to evaluate successively the terms $|\gamma^{\alpha}_{\beta}(\vec{v}_h) - \tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)|_{0,K}$ and $|\rho^{\alpha}_{\beta}(\vec{v}_h) - \tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)|_{0,K}$.

From the definitions (1.1-9)(3.2-8) and relations(1.1-7) (1.1-17), we deduce

$$\gamma^{\alpha}_{\beta}(\vec{v}_h) = a^{\alpha\lambda}\{ \frac{1}{2} (v_{h\beta,\lambda} + v_{h\lambda,\beta}) + \delta^1_{\lambda}\, \delta^1_{\beta}\, R\, v_{h3}\} \tag{4.2-9}$$

$$\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h) = \frac{1}{2}\, a^{\alpha\lambda}_h(\tilde{v}_{h\beta,\lambda} + \tilde{v}_{h\lambda,\beta}) \tag{4.2-10}$$

Now, let us prove the following theorem :

Theorem 4.2-1 : There exists a constant C, independent of h, such that for any functions $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$ et $\vec{v}_h \in \vec{X}_h$ in correspondence through the bijection F_h defined in Theorem 3.2-1, we have

$$|\gamma^{\alpha}_{\beta}(\vec{v}_h) - \tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)|_{0,K} \le Ch\{ \|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{1,K}\}^{1/2} \tag{4.2-11}$$

Proof (in three steps) :

Step 1 : An expression of $\tilde{v}_{h\beta,\lambda}$ as function of the degrees of freedom of $\vec{v}_h$.

For any function $\vec{\tilde{v}}_h$ of $\vec{\tilde{X}}_h$, let us consider its restriction $\vec{\tilde{v}}_h|_K$ to a triangle K of $\mathcal{T}_h$. For simplicity, we shall still denote $\vec{\tilde{v}}_h$ this restriction. By definition, $\tilde{v}_{h\beta} \in P_1(K)$ and thus, we have for any $\xi = (\xi^1,\xi^2) \in K$

$$\tilde{v}_{h\beta}(\xi) = \sum_{i=1}^{3} \lambda_i\, \tilde{v}_{h\beta}(\Sigma_i) \tag{4.2-12}$$

and

$$\tilde{v}_{h\beta,\lambda}(\xi) = \sum_{i=1}^{3} \tilde{v}_{h\beta}(\Sigma_i)\, \frac{\partial\lambda_i}{\partial\xi^{\lambda}} \tag{4.2-13}$$

where we denote $\Sigma_i = (\xi^1_i,\xi^2_i)$, $1 \le i \le 3$, the vertices of the triangle K.

By virtue of compatibility conditions (3.2-10), we have

$$\vec{\tilde{v}}_h(\Sigma_i) = \vec{v}_h(\Sigma_i) \ .$$

Let us denote

$$(4.2\text{-}14) \qquad d_{hk}^{\ j}(\xi) = \vec{a}^j(\xi).\vec{a}_{hk} \quad , \qquad \xi \in K \ .$$

Since by definition $\vec{\tilde{v}}_h(\xi) = \tilde{v}_{hj}(\xi)\vec{a}_h^j$ and $\vec{v}_h(\xi) = v_{hj}(\xi)\vec{a}^j(\xi)$, we obtain

$$(4.2\text{-}15) \qquad \tilde{v}_{h\beta}(\Sigma_i) = d_{h\beta}^{\ j}(\Sigma_i)\ v_{hj}(\Sigma_i) \quad , \ i = 1,\ 2,\ 3 \ .$$

Combining this last relation with (4.2-13), we deduce

$$(4.2\text{-}16) \qquad \tilde{v}_{h\beta,\lambda}(\xi) = \sum_{i=1}^{3} d_{h\beta}^{\ j}(\Sigma_i)\ \frac{\partial\lambda_i}{\partial\xi^\lambda}\, v_{hj}(\Sigma_i) \ .$$

<u>Step 2</u> : <u>Finite expansion of expression</u> (4.2-16)

In the following, we denote C, C_i, C^j, C_i^j ... constants which are independent of h. Thus, we have :

$$(4.2\text{-}17) \quad \left\{ \begin{aligned} \vec{a}^j(\Sigma_i) &= \vec{a}^j(\xi) + (\xi_i^\varepsilon - \xi^\varepsilon)\ \vec{a}^j_{,\varepsilon}(\xi) + 0(h^2)\vec{C}_i^j \ , \\ \vec{a}_j(\Sigma_i) &= \vec{a}_j(\xi) + (\xi_i^\varepsilon - \xi^\varepsilon)\ \vec{a}_{j,\varepsilon}(\xi) + 0(h^2)\vec{C}_{ji} \ . \end{aligned} \right.$$

Using the Gauss and Weingarten relations (see GREEN-ZERNA [1]), i.e.,

$$(4.2\text{-}18) \quad \left\{ \begin{aligned} \vec{a}_{\alpha,\beta} &= \Gamma_{\alpha\beta}^{\nu}\ \vec{a}_\nu + b_{\alpha\beta}\ \vec{a}_3 && \\ \vec{a}^{\alpha}_{,\beta} &= -\ \Gamma_{\beta\nu}^{\alpha}\ \vec{a}^\nu + b_\beta^\alpha\ \vec{a}^3 && \text{(Gauss)} \\ \vec{a}_{3,\alpha} &= \vec{a}^3_{,\alpha} = -b_\alpha^\lambda\ \vec{a}_\lambda && \text{(Weingarten)} \end{aligned} \right.$$

we deduce from (4.2-17), by virtue of (1.1-7)(1.1-8) and (1.1-14)

$$(4.2\text{-}19)\quad \begin{cases} \vec{a}^{\alpha}(\Sigma_i) = \vec{a}^{\alpha}(\xi) - \frac{1}{R}\,(\xi_i^1-\xi^1)\ \delta_1^{\alpha}\ \vec{a}_3(\xi) + O(h^2)\vec{C}_i^{\alpha}\ , \\ \vec{a}_{\alpha}(\Sigma_i) = \vec{a}_{\alpha}(\xi) - R(\xi_i^1-\xi^1)\ \delta_{\alpha}^1\ \vec{a}_3(\xi) + O(h^2)\vec{C}_{\alpha i}\ , \\ \vec{a}_3(\Sigma_i) = \vec{a}_3(\xi) + \frac{1}{R}\,(\xi_i^1-\xi^1)\ \vec{a}_1(\xi) + O(h^2)\vec{C}_i\ . \end{cases}$$

By definition of the mapping $\vec{\phi}_h$, we have at any point $\xi \in K$

$$\vec{\phi}_h(\xi) = \sum_{i=1}^{3} \lambda_i\ \vec{\phi}_h(\Sigma_i) = \sum_{i=1}^{3} \lambda_i\ \vec{\phi}(\Sigma_i)\ ,$$

and thus

$$\vec{a}_{h\beta} = \vec{\phi}_{h,\beta}(\xi) = \sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^{\beta}}\ \vec{\phi}(\Sigma_i)\ .$$

But,

$$\vec{\phi}(\Sigma_i) = \vec{\phi}(\xi) + (\xi_i^{\varepsilon}-\xi^{\varepsilon})\ \frac{\partial\vec{\phi}}{\partial\xi^{\varepsilon}}\,(\xi) +$$

$$+ \frac{1}{2}\,(\xi_i^{\varepsilon}-\xi^{\varepsilon})(\xi_i^{\eta}-\xi^{\eta})\ \frac{\partial^2\vec{\phi}}{\partial\xi^{\varepsilon}\partial\xi^{\eta}}\,(\xi) + O(h^3)\vec{C}_i\ ,\ \forall\xi \in K\ .$$

Noticing that

$$(4.2\text{-}20)\qquad \sum_{i=1}^{3} \lambda_i = 1\quad ,\quad \sum_{i=1}^{3} \lambda_i(\xi_i^{\varepsilon}-\xi^{\varepsilon}) = 0\quad ,\quad \sum_{i=1}^{3} (\xi_i^{\varepsilon}-\xi^{\varepsilon})\ \frac{\partial\lambda_i}{\partial\xi^{\beta}} = \delta_{\beta}^{\varepsilon}$$

and that

$$\frac{\partial\lambda_i}{\partial\xi^{\beta}} = O(h^{-1})C_{\beta}\quad ,$$

we obtain by combining (4.2-18) with (1.1-7)

$$(4.2\text{-}21)\qquad \vec{a}_{h\beta} = \vec{a}_{\beta}(\xi) - \frac{1}{2}\,R\,\sum_{j=1}^{3} \frac{\partial\lambda_j}{\partial\xi^{\beta}}\,(\xi_j^1-\xi^1)^2\ \vec{a}_3(\xi) + O(h^2)\vec{C}_{\beta}\ ,\ \forall\xi \in K\ .$$

Then, from (1.1-3)(4.2-14)(4.2-19) and (4.2-21), we derive

$$(4.2\text{-}22)\quad \begin{cases} d^{\alpha}_{h\beta}(\Sigma_i) = \delta^{\alpha}_{\beta} + O(h^2)C^{\alpha}_{\beta i}, \\ d^{3}_{h\beta}(\Sigma_i) = R\{(\xi^1_i-\xi^1)\ \delta^1_{\beta} - \frac{1}{2}\sum_{j=1}^{3} \frac{\partial\lambda_j}{\partial\xi^{\beta}}(\xi^1_j-\xi^1)^2\} + O(h^2)C_{\beta i},\ \forall\xi \in K\ . \end{cases}$$

Finally, since $v_{h\alpha} \in P_1(K)$, $v_{h3} \in P_3(K_i)$ (K_i = subtriangle of K, $\overline{K} = \bigcup_{i=1}^{3} \overline{K}_i$) and $v_{h3} \in \mathscr{C}^1(K)$, we have for any $\xi \in K$:

$$(4.2\text{-}23)\quad \begin{cases} v_{h\alpha}(\Sigma_i) = v_{h\alpha}(\xi) + (\xi^{\varepsilon}_i-\xi^{\varepsilon})\ v_{h\alpha,\varepsilon} \\ v_{h3}(\Sigma_i) = v_{h3}(\xi) + (\xi^{\varepsilon}_i-\xi^{\varepsilon})\ v_{h3,\varepsilon}(\overline{\xi}_i)\ ,\ \text{where } \overline{\xi}_i \in [\Sigma_i,\xi]. \end{cases}$$

Combining finite expansions (4.2-22)(4.2-23) with the expression (4.2-16) and using relations (4.2-20), we deduce :

$$\begin{aligned} \tilde{v}_{h\beta,\lambda}(\xi) &= v_{h\beta,\lambda}(\xi) + R\ \delta^1_{\beta}\ \delta^1_{\lambda}\ v_{h3}(\xi) + \\ &+ O(h)\ \{C^j_{\lambda\beta}v_{hj}(\xi) + O(h)C^{\alpha\varepsilon}_{\beta\lambda}v_{h\alpha,\varepsilon} + \sum_{i=1}^{3} C^{\varepsilon}_{\lambda\beta i}\ v_{h3,\varepsilon}(\overline{\xi}_i)\}\ . \end{aligned}$$

Substituting the previous finite expansion into the expression (4.2-10) and using (4.2-21), we finally prove that :

$$\begin{cases} \tilde{\gamma}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h) = \frac{1}{2}\ a^{\alpha\lambda}(\xi)\{v_{h\beta,\lambda}(\xi) + v_{h\lambda,\beta}(\xi)\} + \frac{1}{R}\ \delta^{\alpha}_1\ \delta^1_{\beta}\ v_{h3}(\xi) + \\ \qquad + O(h)\ \{C^{\alpha j}_{\beta}\ v_{hj}(\xi) + O(h)C^{\alpha\nu\varepsilon}_{\beta}v_{h\nu,\varepsilon} + \sum_{i=1}^{3} C^{\alpha\varepsilon}_{\beta i}\ v_{h3,\varepsilon}(\overline{\xi}_i)\} \end{cases}$$

or, using the definition (4.2-9)

$$(4.2\text{-}24)\quad \begin{cases} \tilde{\gamma}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h) = \gamma^{\alpha}_{\beta}(\vec{v}_h) + O(h)\ \{C^{\alpha j}_{\beta}\ v_{hj}(\xi) + \\ \qquad + O(h)\ C^{\alpha\nu\varepsilon}_{\beta}v_{h\nu,\varepsilon} + \sum_{i=1}^{3} C^{\alpha\varepsilon}_{\beta i}\ v_{h3,\varepsilon}(\overline{\xi}_i)\}\ . \end{cases}$$

Step 3 : Obtention of the estimate (4.2-11)

Taking into account that

$$\text{meas}(K) = O(h^2)$$

we deduce from finite expansion (4.2-24) that there exists a constant C independent of h such that for any $K \in \mathscr{T}_h$

$$(4.2\text{-}25) \qquad |\tilde{\gamma}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h) - \gamma^{\alpha}_{\beta}(\vec{v}_h)|_{0,K} \le Ch^2 \{ \sum_{j=1}^{3} |v_{hj}|^2_{0,\infty,K} + \sum_{j=1}^{3} |v_{hj}|^2_{1,\infty,K} \}^{1/2} .$$

To conclude we need following additional results :

Lemma 4.2-1 : Let k be a given integer. There exists a constant C, independent of $\hat{v} \in P_k$, such that

$$|\hat{v}|_{j,\hat{K}} \le C|\hat{v}|_{i,\hat{K}} \quad , \quad 0 \le i \le j \quad , \quad \forall \hat{v} \in P_k \quad ,$$

$$|\hat{v}|_{j,\infty,\hat{K}} \le C|\hat{v}|_{j,\hat{K}} \quad , \quad j \ge 0 \quad , \quad \forall \hat{v} \in P_k \quad ,$$

where $\hat{K}$ is a reference triangle (for example, the triangle of vertices $\hat{a}_1 = (0,0)$, $\hat{a}_2 = (0,1)$, $\hat{a}_3 = (1,0)$) and where P_k means the space of all polynomials of degree $\le k$.

□

Using Lemma 4.2-1 and results of interpolation theory in Sobolev spaces - see CIARLET [1, Theorems 3.1-2 and 3.1-3] , we deduce

$$|v_{hj}|_{p,\infty,K} \le Ch^{-1} \; |v_{hj}|_{p,K} \quad , \quad p = 0,1 \; ,$$

so that relation (4.2-25) gives the estimate (4.2-11).

□

Now, we have to estimate $|\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h) - \rho^{\alpha}_{\beta}(\vec{v}_h)|_{0,K}$. By virtue of definitions (1.1-16)(1.1-17) and (3.2-9), we obtain :

$$(4.2\text{-}26)\quad \begin{cases} \rho^{\alpha}_{\beta}(\vec{v}_h) = a^{\alpha\lambda}\ \rho_{\lambda\beta}(\vec{v}_h)\ , \\ \text{with } \rho_{\lambda\beta}(\vec{v}_h) = v_{h3,\lambda\beta} - \frac{1}{4R}\ \delta^1_{\lambda}(3v_{h1,\beta} - v_{h\beta,1}) - \frac{1}{4R}\ \delta^1_{\beta}(3v_{h1,\lambda} - v_{h\lambda,1}), \end{cases}$$

$$(4.2\text{-}27)\quad \begin{cases} \tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h) = a^{\alpha\lambda}_h\ \tilde{\rho}_{h\lambda\beta}(\vec{\tilde{v}}_h)\ , \\ \text{with } \tilde{\rho}_{h\lambda\beta}(\vec{\tilde{v}}_h) = \tilde{v}_{h3,\lambda\beta}. \end{cases}$$

In what follows (precisely, in the last part of the proof of Theorem 4.2-2), we shall assume that the families of triangulations $\mathcal{T}_h$ satisfy, besides the regularity hypothesis (2.1-1)(2.1-2), the following property :

(4.2-28) The triangulations $\mathcal{T}_h$ are constructed so that any triangle of $\mathcal{T}_h$ has one of its edges parallel to the cylinder axis, i.e., one of its edges is supported by a line ξ^1 = constant.

An example of such a triangulation of Ω is indicated on Figure 4.2-1.

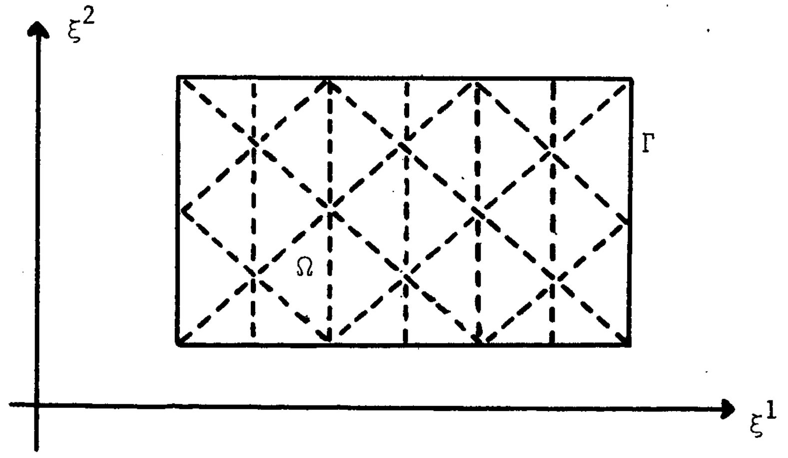


Figure 4.2-1 : The choice of the triangulation $\mathcal{T}_h$ of Ω.

Let us prove the following theorem :

Theorem 4.2-2 : Under the Hypothesis (4.2-28), there exists a constant C, independent of h, such that for any $\vec{v}_h \in \vec{X}_h$ and any $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$ in correspondence through the bijection F_h defined in Theorem 3.2-1, we have

$$(4.2\text{-}29) \qquad |\rho^{\alpha}_{\beta}(\vec{v}_h) - \tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)|_{0,K} \leq Ch\{ \|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{2,K}\}^{1/2} .$$

Proof (in six steps) :

Step 1 : Expression of $\tilde{v}_{h3,\alpha\beta}$ as function of degrees of freedom of $\vec{\tilde{X}}_h$.

By definition of space $\tilde{X}_{h2}$, we have on any subtriangle K_j , $j = 1,2,3$, of a given triangle K with vertices $\Sigma_i = (\xi^1_i,\xi^2_i)$, $i = 1,2,3$,

$$\left\{\begin{aligned} \tilde{v}_{h3j}(\xi) = &\sum_{i=1}^{3} \tilde{v}_{h3}(\Sigma_i)\, p^o_{j,i}(\lambda) + \sum_{i=1}^{3} D\tilde{v}_{h3}(\Sigma_i)(\Sigma_{i+1}-\Sigma_i)\, p^1_{j,i,i+1}(\lambda) + \\ &+ \sum_{i=1}^{3} D\tilde{v}_{h3}(\Sigma_i)(\Sigma_{i-1}-\Sigma_i)\, p^1_{j,i,i-1}(\lambda) , \end{aligned}\right.$$

where $p^o_{j,i}$, $p^1_{j,i,i+1}$, $p^1_{j,i,i-1}$, $1 \leq i \leq 3$, denote the basis functions for the subtriangle K_j of the reduced finite element of HSIEH-CLOUGH-TOCHER - see BERNADOU-HASSAN [1] - and where $\tilde{v}_{h3j}$ denotes the restriction of function $\tilde{v}_{h3}$ to subtriangle K_j. The above expression can be rewritten as

$$\left\{\begin{aligned} \tilde{v}_{h3j}(\xi) = &\sum_{i=1}^{3} \tilde{v}_{h3}(\Sigma_i)\, p^o_{j,i}(\lambda) + \sum_{i=1}^{3} (\xi^{\nu}_{i+1}-\xi^{\nu}_i)\tilde{v}_{h3,\nu}(\Sigma_i)\, p^1_{j,i,i+1}(\lambda) + \\ &+ \sum_{i=1}^{3} (\xi^{\nu}_{i-1}-\xi^{\nu}_i)\tilde{v}_{h3,\nu}(\Sigma_i)\, p^1_{j,i,i-1}(\lambda) , \end{aligned}\right.$$

so that

$$
(4.2\text{-}30)\left\{
\begin{aligned}
\tilde{v}_{h3j,\alpha\beta}(\xi) = &\sum_{i=1}^{3} \tilde{v}_{h3}(\Sigma_i)(p^o_{j,i}(\lambda))_{,\alpha\beta} + \\
&+ \sum_{i=1}^{3} (\xi^\nu_{i+1}-\xi^\nu_i)\tilde{v}_{h3,\nu}(\Sigma_i)(p^1_{j,i,i+1}(\lambda))_{,\alpha\beta} + \\
&+ \sum_{i=1}^{3} (\xi^\nu_{i-1}-\xi^\nu_i)\tilde{v}_{h3,\nu}(\Sigma_i)(p^1_{j,i,i-1}(\lambda))_{,\alpha\beta}
\end{aligned}
\right.
$$

<u>Step 2</u> : <u>Expressions of $\tilde{v}_{h3}(\Sigma_i)$ and $\tilde{v}_{h3,\nu}(\Sigma_i)$ as function of degrees of freedom of space</u> $\vec{X}_h$

First, by virtue of compatibility relation (3.2-10), we deduce with (4.2-14)

$$(4.2\text{-}31)\qquad \tilde{v}_{h3}(\Sigma_i) = d^{\,j}_{h3}(\Sigma_i)\; v_{hj}(\Sigma_i)$$

Next, from (3.2-22), we deduce

$$(4.2\text{-}32)\qquad \tilde{v}_{h3,\nu}(\Sigma_i) = e_{\lambda\nu}\,\sqrt{a_h}\;\tilde{\omega}^\lambda_h(\Sigma_i)\ .$$

Equations (3.2-21) show that $\tilde{\omega}^\lambda_h(\Sigma_i)$ are the unique solutions of the system

$$(4.2\text{-}33)\qquad d^{\,\lambda}_{h\alpha}(\Sigma_i)\;\tilde{\omega}^\alpha_h(\Sigma_i) = B^\lambda(\Sigma_i) - d^{\,\lambda}_{h3}(\Sigma_i)\;\tilde{\omega}^3_h(\Sigma_i)\ .$$

Let us remark that $\tilde{\omega}^3_h(\Sigma_i)$ can be written as function of $\vec{v}_h$; indeed, relations (3.2-20) and (4.2-16) yield

$$(4.2\text{-}34)\qquad \tilde{\omega}^3_h(\Sigma_i) = \frac{1}{2\sqrt{a_h}}\,e^{\nu\beta}\sum_{k=1}^{3} d^{\,j}_{h\beta}(\Sigma_k)\,\frac{\partial\lambda_k}{\partial\xi^\nu}\,v_{hj}(\Sigma_k)\ .$$

Afterwards, from (3.2-18), we have

$$(4.2\text{-}35)\qquad B^\lambda(\Sigma_i) = \frac{1}{\sqrt{a(\Sigma_i)}}\,e^{\lambda\beta}\;\{v_{h3,\beta}(\Sigma_i) + b^\alpha_\beta(\Sigma_i)v_{h\alpha}(\Sigma_i)\}\ .$$

From equations (4.2-32)(4.2-33), we deduce that $\tilde{v}_{h3,\nu}(\Sigma_i)$ are solutions of the system

$$(4.2\text{-}36)\quad \left\{ \begin{aligned} &-\tilde{v}_{h3,1}(\Sigma_i)\, d_{h2}^{\lambda}(\Sigma_i) + \tilde{v}_{h3,2}(\Sigma_i)\, d_{h1}^{\lambda}(\Sigma_i) = \\ &\qquad = \sqrt{a_h}\, \{B^{\lambda}(\Sigma_i) - d_{h3}^{\lambda}(\Sigma_i)\, \tilde{\omega}_h^3(\Sigma_i)\} . \end{aligned} \right.$$

From (1.1-8), expressions (4.2-35) give

$$\left\{ \begin{aligned} B^1(\Sigma_i) &= \frac{1}{\sqrt{a(\Sigma_i)}}\, v_{h3,2}(\Sigma_i) \\ B^2(\Sigma_i) &= \frac{-1}{\sqrt{a(\Sigma_i)}}\, \{v_{h3,1}(\Sigma_i) - \frac{1}{R}\, v_{h1}(\Sigma_i)\} . \end{aligned} \right.$$

Then, if we denote

$$(4.2\text{-}37)\quad d_h(\xi) = d_{h1}^1(\xi)\, d_{h2}^2(\xi) - d_{h1}^2(\xi)\, d_{h2}^1(\xi) ,$$

we derive from (4.2-36)

$$(4.2\text{-}38)\quad \left\{ \begin{aligned} &\tilde{v}_{h3,\nu}(\Sigma_i) = \\ &\quad \frac{\sqrt{a_h}}{d_h(\Sigma_i)\sqrt{a(\Sigma_i)}}\, \{d_{h\nu}^1(\Sigma_i)[v_{h3,1}(\Sigma_i) - \frac{1}{R}\, v_{h1}(\Sigma_i)] + d_{h\nu}^2(\Sigma_i) v_{h3,2}(\Sigma_i)\} - \\ &- \frac{1}{2}\, \frac{1}{d_h(\Sigma_i)}\, \{[d_{h\nu}^2(\Sigma_i) d_{h3}^1(\Sigma_i) - d_{h\nu}^1(\Sigma_i) d_{h3}^2(\Sigma_i)] \sum_{k=1}^{3} [d_{h2}^j(\Sigma_k)\, \frac{\partial\lambda_k}{\partial\xi^1} - \\ &\qquad\qquad - d_{h1}^j(\Sigma_k)\, \frac{\partial\lambda_k}{\partial\xi^2}]\, v_{hj}(\Sigma_k)\} \end{aligned} \right.$$

<u>Step 3</u> : <u>Finite expansion of</u> $\tilde{v}_{h3}(\Sigma_i)$

We have

$$\vec{a}^j(\Sigma_i) = \vec{a}^j(\xi) + (\xi_i^{\varepsilon} - \xi^{\varepsilon})\, \vec{a}^j_{,\varepsilon}(\xi) + \frac{1}{2}(\xi_i^{\varepsilon} - \xi^{\varepsilon})(\xi_i^{\eta} - \xi^{\eta}) \vec{a}^j_{,\varepsilon\eta}(\xi) + O(h^3)\vec{C}^j ,$$

where, by denoting

$$\Gamma^3_{\alpha\beta} = b_{\alpha\beta} \quad , \quad \Gamma^\beta_{3\alpha} = - b^\beta_\alpha \quad , \quad \Gamma^3_{3\alpha} = 0 \ ,$$

and by using (4.2-18), the derivatives $\vec{a}^j_{,\varepsilon}$ and $\vec{a}^j_{,\varepsilon\eta}$ are given by

$$\left\{ \begin{aligned} \vec{a}^j_{,\varepsilon} &= - \Gamma^j_{k\varepsilon} \vec{a}^k \\ \vec{a}^j_{,\varepsilon\eta} &= \Gamma^j_{i\varepsilon} \Gamma^i_{k\eta} \vec{a}^k - \Gamma^j_{k\varepsilon,\eta} \vec{a}^k \end{aligned} \right.$$

i.e., by using (1.1-7), (1.1-8) and (1.1-14) ,

$$\left\{ \begin{aligned} \vec{a}^1_{,\varepsilon} &= - \frac{1}{R} \delta^1_\varepsilon \vec{a}^3 \\ \vec{a}^2_{,\varepsilon} &= 0 \\ \vec{a}^3_{,\varepsilon} &= \frac{1}{R} \delta^1_\varepsilon \vec{a}_1 \end{aligned} \right.$$

and

$$\left\{ \begin{aligned} \vec{a}^1_{,\varepsilon\eta} &= - \delta^1_\varepsilon \delta^1_\eta \vec{a}^1 \\ \vec{a}^2_{,\varepsilon\eta} &= 0 \\ \vec{a}^3_{,\varepsilon\eta} &= - \delta^1_\varepsilon \delta^1_\eta \vec{a}_3 \end{aligned} \right.$$

Whence

$$(4.2\text{-}39) \quad \left\{ \begin{aligned} \vec{a}^1(\Sigma_i) &= \vec{a}^1(\xi) - \frac{1}{R} (\xi^1_i - \xi^1) \vec{a}^3(\xi) - \frac{1}{2} (\xi^1_i - \xi^1)^2 \vec{a}^1(\xi) + 0(h^3)\vec{C}^1 \\ \vec{a}^2(\Sigma_i) &= \vec{a}^2(\xi) + 0(h^3)\vec{C}^2 \\ \vec{a}_3(\Sigma_i) &= \vec{a}_3(\xi) + \frac{1}{R} (\xi^1_i - \xi^1) \vec{a}_1(\xi) - \frac{1}{2} (\xi^1_i - \xi^1)^2 \vec{a}_3(\xi) + 0(h^3)\vec{C}^3 \end{aligned} \right.$$

Likewise, we have

$$\left\{\begin{aligned}\vec{\phi}(\Sigma_i) &= \vec{\phi}(\xi) + (\xi_i^\varepsilon-\xi^\varepsilon)\vec{a}_\varepsilon(\xi) + \frac{1}{2}(\xi_i^\varepsilon-\xi^\varepsilon)(\xi_i^\eta-\xi^\eta)\vec{a}_{\varepsilon,\eta}(\xi) + \\ &+ \frac{1}{6}(\xi_i^\varepsilon-\xi^\varepsilon)(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda)\vec{a}_{\varepsilon,\eta\lambda}(\xi) + O(h^4) .\end{aligned}\right.$$

Since $\vec{a}_{h\beta} = \vec{\phi}_{h,\beta}(\xi) = \sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^\beta}\vec{\phi}(\Sigma_i)$, we get for the circular cylinder

$$\left\{\begin{aligned}\vec{a}_{h\beta} &= \vec{a}_\beta(\xi) + \frac{1}{2}\Gamma^k_{\varepsilon\eta}\sum_{i=1}^{3}\frac{\partial\lambda_i}{\partial\xi^\beta}(\xi_i^\varepsilon-\xi^\varepsilon)(\xi_i^\eta-\xi^\eta)\vec{a}_k(\xi) + \\ &+ \frac{1}{6}\Gamma^j_{\varepsilon\eta}\Gamma^k_{j\lambda}\sum_{i=1}^{3}\frac{\partial\lambda_i}{\partial\xi^\beta}(\xi_i^\varepsilon-\xi^\varepsilon)(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda)\vec{a}_k(\xi) + O(h^3)\vec{C}_\beta .\end{aligned}\right.$$

By using again (1.1-7), (1.1-8) and (1.1-14), we deduce

$$\text{(4.2-40)}\quad\left\{\begin{aligned}\vec{a}_{h\beta} &= \vec{a}_\beta(\xi) - R\,A_\beta(\xi^1)\vec{a}_3(\xi) - B_\beta(\xi^1)\vec{a}_1(\xi) + O(h^3)\vec{C}_\beta ,\\ \vec{a}_{h3} &= \frac{\sqrt{a(\xi)}}{\sqrt{a_h}}\{\vec{a}_3(\xi) + R\,A_\lambda(\xi^1)\vec{a}^\lambda(\xi) - B_1(\xi^1)\vec{a}_3(\xi) + O(h^3)\vec{C}_3\} ,\end{aligned}\right.$$

where

$$\text{(4.2-41)}\quad\left\{\begin{aligned}A_\beta(\xi^1) &= \frac{1}{2}\sum_{j=1}^{3}\frac{\partial\lambda_j}{\partial\xi^\beta}(\xi_j^1-\xi^1)^2 ,\\ B_\beta(\xi^1) &= \frac{1}{6}\sum_{j=1}^{3}\frac{\partial\lambda_j}{\partial\xi^\beta}(\xi_j^1-\xi^1)^3 .\end{aligned}\right.$$

Consequently,

$$\text{(4.2-42)}\quad a_h = a(\xi)(1-2\,B_1(\xi^1)) + R^2(A_1(\xi^1))^2 + R^4(A_2(\xi^1))^2 + O(h^3) ,$$

and thus

$$\text{(4.2-43)}\quad\left\{\begin{aligned}\vec{a}_{h3} &= \vec{a}_3(\xi) + R\,A_\lambda(\xi^1)\vec{a}^\lambda(\xi) - \frac{1}{2}(A_1(\xi^1))^2\vec{a}_3(\xi) - \\ &- \frac{1}{2}R^2(A_2(\xi^1))^2\vec{a}_3(\xi) + O(h^3)\vec{C}_3 .\end{aligned}\right.$$

By combining (4.2-14), (4.2-39), (4.2-40) and (4.2-43), we find

$$(4.2\text{-}44)\quad \begin{cases} d_{h3}^{\,1}(\Sigma_i) = \frac{1}{R}\,\{A_1(\xi^1) - (\xi_i^1-\xi^1)\} + O(h^3)C_3^1 \\ d_{h3}^{\,2}(\Sigma_i) = R\,A_2(\xi^1) + O(h^3)C_3^2 \\ d_{h3}^{\,3}(\Sigma_i) = 1 + A_1(\xi^1)(\xi_i^1-\xi^1) - \frac{1}{2}\,(\xi_i^1-\xi^1)^2 - \frac{1}{2}\,(A_1(\xi^1))^2 - \\ \qquad\qquad - \frac{1}{2}\,R^2(A_2(\xi^1))^2 + O(h^3)C_3^3 \end{cases}$$

With above finite expansions, relation (4.2-31) yields

$$(4.2\text{-}45)\quad \begin{cases} \tilde{v}_{h3}(\Sigma_i) = \frac{1}{R}\,\{A_1(\xi^1) - (\xi_i^1-\xi^1)\}\;v_{h1}(\Sigma_i) + R\,A_2(\xi^1)\;v_{h2}(\Sigma_i) + \\ \qquad + \{1 + A_1(\xi^1)(\xi_i^1-\xi^1) - \frac{1}{2}\,(\xi_i^1-\xi^1)^2 - \frac{1}{2}\,(A_1(\xi^1))^2 - \\ \qquad\qquad - \frac{1}{2}\,R^2(A_2(\xi^1))^2\}\;v_{h3}(\Sigma_i) + O(h^3)\;\{C_3^j\;v_{hj}(\Sigma_i)\} \end{cases}$$

<u>Step 4</u> : <u>Finite expansion of</u> $\tilde{v}_{h3,\nu}(\Sigma_i)$.

From (4.2-22) and (4.2-37), we have

$$(4.2\text{-}46)\qquad d_h(\Sigma_i) = 1 + O(h^2)$$

and from (4.2-19), we obtain

$$(4.2\text{-}47)\qquad a(\Sigma_i) = a(\xi) + O(h^2)$$

Combining finite expansions (4.2-22), (4.2-23), (4.2-42), (4.2-44), (4.2-46), (4.2-47) with relation (4.2-38), we get the existence of constants C_ν, C_ν^ε, $C_{\nu\eta}^\varepsilon$, $C_{3\nu}^\varepsilon$ independent of h, such that

$$
(4.2\text{-}48)\left\{
\begin{aligned}
&\tilde{v}_{h3,\nu}(\Sigma_i) = v_{h3,\nu}(\Sigma_i) - \frac{1}{R}\,\delta^1_\nu\, v_{h1}(\Sigma_i) + \\
&\qquad + \frac{1}{2}\,\{v_{h1,2}(\xi)-v_{h2,1}(\xi)\}\,\{\frac{1}{R}\,e_{1\nu}[A_1(\xi^1)-(\xi^1_i-\xi^1)] + Re_{2\nu}\,A_2(\xi^1)\} + \\
&\qquad + 0(h^2)\;\{C^\varepsilon_{3\nu}v_{h3,\varepsilon}(\Sigma_i) + \sum_{k=1}^{3} C^j_\nu v_{hj}(\Sigma_k) + C^\varepsilon_\nu v_{h3,\varepsilon}(\bar{\xi}_k)\}\ .
\end{aligned}\right.
$$

Step 5 : Finite expansion of $\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)$

This finite expansion is obtained by substituting (4.2-45) and (4.2-48) into the expression (4.2-30). Then, we get the existence of constants C independent of h such that at any point ξ of any subtriangle K_j , $j = 1,2,3$, of the triangle K with vertices Σ_i , $i = 1,2,3$, we have

$$
(4.2\text{-}49)\left\{
\begin{aligned}
&\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)\Big|_{K_j} = \sum_{i=1}^{3}\{\,\frac{1}{R}\,[A_1(\xi^1)-(\xi^1_i-\xi^1)]v_{h1}(\Sigma_i) + R\,A_2(\xi^1)v_{h2}(\Sigma_i) + \\
&\qquad + [1 + A_1(\xi^1)(\xi^1_i-\xi^1) - \frac{1}{2}\,(\xi^1_i-\xi^1)^2 - \frac{1}{2}\,(A_1(\xi^1))^2 - \\
&\qquad\qquad\qquad - \frac{1}{2}\,R^2(A_2(\xi^1))^2]v_{h3}(\Sigma_i)\}\,(p^o_{j,i})_{,\alpha\beta} + \\
&\qquad + \sum_{i=1}^{3}\{v_{h3,\nu}(\Sigma_i) - \frac{1}{R}\,\delta^1_\nu\,v_{h1}(\Sigma_i) + \\
&\qquad + \frac{1}{2}\,(v_{h1,2}(\xi)-v_{h2,1}(\xi))\,\{\frac{1}{R}\,e_{1\nu}[A_1(\xi^1)-(\xi^1_i-\xi^1)] + Re_{2\nu}\,A_2(\xi^1)\}\} \times \\
&\qquad \times \{(\xi^\nu_{i+1}-\xi^\nu_i)(p^1_{j,i,i+1})_{,\alpha\beta} + (\xi^\nu_{i-1}-\xi^\nu_i)(p^1_{j,i,i-1})_{,\alpha\beta}\} \\
&\qquad + 0(h)\,\sum_{k=1}^{3}[C^j_{\alpha\beta}v_{hj}(\Sigma_k) + C^\varepsilon_{\alpha\beta}v_{h3,\varepsilon}(\Sigma_k) + C^\varepsilon_{\alpha\beta}v_{h3,\varepsilon}(\bar{\xi}_k)]\}\ ,\ \bar{\xi}_k \in [\xi,\Sigma_k]
\end{aligned}\right.
$$

By virtue of hypothesis (4.2-28), we can assume that we study this finite expansion in a neighbouring of a point ξ of a subtriangle K_j, $j = 1,2,3$, of a triangle K as represented in Figure 4.2.2.

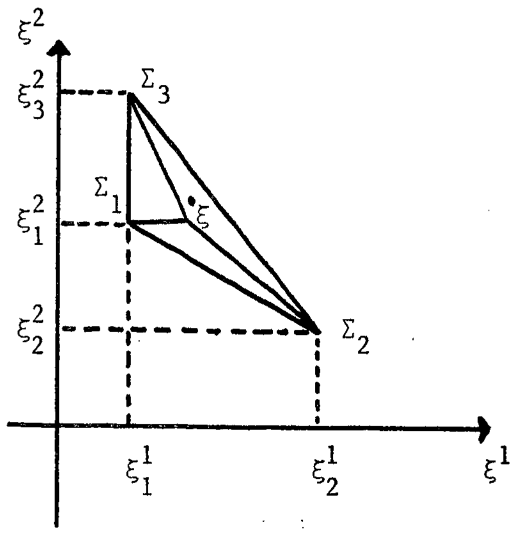


Figure 4.2-2

From (3.2-14), we have for such a triangle

$$(4.2\text{-}50)\quad \begin{cases} A_1(\xi^1) = \frac{1}{2}\{(\xi_1^1-\xi^1) + (\xi_2^1-\xi^1)\} , \\ A_2(\xi^1) = 0 \quad . \end{cases}$$

Besides, let us remark that

$$(4.2\text{-}51)\quad A_1(\xi^1) - (\xi_i^1-\xi^1) = A_1(\Sigma_i) \quad .$$

Now, we are going to study the finite expansion (4.2-49) by taking into account results (4.2-50) and (4.2-51).

(A) First, <u>let us examine the terms in</u> $v_{h3}(\Sigma_i)$ <u>and</u> $v_{h3,\nu}(\Sigma_i)$

By definition of space X_{h2}, we have (refer to Step 1 and (4.2-30)) :

$$(4.2\text{-}52)\quad \begin{cases} v_{h3j,\alpha\beta}(\xi) = \sum\limits_{i=1}^{3} v_{h3}(\Sigma_i)(p_{j,i}^{o})_{,\alpha\beta} + \\ \quad + \sum\limits_{i=1}^{3} \{(\xi_{i+1}^{\nu}-\xi_i^{\nu})(p_{j,i,i+1}^{1})_{,\alpha\beta} + (\xi_{i-1}^{\nu}-\xi_i^{\nu})(p_{j,i,i-1}^{1})_{,\alpha\beta}\}v_{h3,\nu}(\Sigma_i). \end{cases}$$

Next, since $A_1(\xi^1)(\xi_i^1-\xi^1) - \frac{1}{2}(\xi_i^1-\xi^1)^2 = \frac{1}{2}(\xi_1^1-\xi^1)(\xi_2^1-\xi^1)$, we obtain with (4.2-23)

$$\left\{\begin{aligned}&\sum_{i=1}^{3}\{A_1(\xi^1)(\xi_i^1-\xi^1) - \frac{1}{2}(\xi_i^1-\xi^1)^2\}\, v_{h3}(\Sigma_i)(p_{j,i}^o)_{,\alpha\beta} = \\ &\qquad = \frac{1}{2}(\xi_1^1-\xi^1)(\xi_2^1-\xi^1)\sum_{i=1}^{3}\{v_{h3}(\xi)+(\xi_i^\varepsilon-\xi^\varepsilon)v_{h3,\varepsilon}(\bar{\xi}_i)\}(p_{j,i}^o)_{,\alpha\beta}\end{aligned}\right.$$

with $\bar{\xi}_i \in [\Sigma_i,\xi]$.

But, from (4.2-52) we get $\sum_{i=1}^{3}(p_{j,i}^o)_{,\alpha\beta} = 0$, and thus

$$(4.2\text{-}53)\quad \sum_{i=1}^{3}\{A_1(\xi^1)(\xi_i^1-\xi^1) - \frac{1}{2}(\xi_i^1-\xi^1)^2\}\, v_{h3}(\Sigma_i)(p_{j,i}^o)_{,\alpha\beta} = O(h)\sum_{i=1}^{3} v_{h3,\varepsilon}(\bar{\xi}_i).$$

In the same way,

$$(4.2\text{-}54)\quad \frac{1}{2}(A_1(\xi^1))^2\sum_{i=1}^{3} v_{h3}(\Sigma_i)(p_{j,i}^o)_{,\alpha\beta} = O(h)\sum_{i=1}^{3} v_{h3,\varepsilon}(\bar{\xi}_i)\ .$$

B) Now, <u>let us examine the terms in</u> $v_{h\lambda}(\Sigma_i)$ and $v_{h\lambda,\varepsilon}(\Sigma_i)$

Let us consider the function $A_1(\xi^1)\, v_{h1}(\xi)$. We have

$$(A_1(\xi^1)v_{h1})_{,\alpha} = -\, v_{h1}\,\delta_\alpha^1 + A_1(\xi^1)v_{h1,\alpha}$$

and

$$(A_1(\xi^1)v_{h1})_{,\alpha\beta} = -\, v_{h1,\beta}\,\delta_\alpha^1 - v_{h1,\alpha}\,\delta_\beta^1\ .$$

Thus, since $A_1(\xi^1)v_{h1}$ is invariant by HCT-R interpolation, we get from (4.2-52)

$$(4.2\text{-}55)\quad \left\{\begin{aligned} &\sum_{i=1}^{3} A_1(\Sigma_i)v_{h1}(\Sigma_i)(p^{o}_{j,i})_{,\alpha\beta} + \\ &+ \sum_{i=1}^{3} [A_1(\Sigma_i)v_{h1,1}(\Sigma_i)-v_{h1}(\Sigma_i)][(\xi^1_{i+1}-\xi^1_i)(p^1_{j,i,i+1})_{,\alpha\beta} + \\ &\qquad\qquad + (\xi^1_{i-1}-\xi^1_i)(p^1_{j,i,i-1})_{,\alpha\beta}] + \\ &+ \sum_{i=1}^{3} A_1(\Sigma_i)v_{h1,2}(\Sigma_i)[(\xi^2_{i+1}-\xi^2_i)(p^1_{j,i,i+1})_{,\alpha\beta} + \\ &\qquad\qquad + (\xi^2_{i-1}-\xi^2_i)(p^1_{j,i,i-1})_{,\alpha\beta}] = \\ &= - 2v_{h1,1}(\xi)\, \delta^1_\alpha\, \delta^1_\beta - v_{h1,2}(\xi)(\delta^1_\alpha\, \delta^2_\beta + \delta^2_\alpha\, \delta^1_\beta) \end{aligned}\right.$$

Afterwards, let us denote $f(\xi) = -\frac{1}{2}\{(\xi^1)^2 - \sum_{j=1}^{3} \lambda_j(\xi)(\xi^1_j)^2\}$. We have :

$$\left\{\begin{aligned} &f(\Sigma_i) = 0 \\ &f_{,1}(\Sigma_i) = A_1(\Sigma_i) \\ &f_{,2}(\Sigma_i) = 0 \\ &f_{,\alpha\beta}(\xi) = -\delta^1_\alpha\, \delta^1_\beta \end{aligned}\right.$$

and thus, since $v_{h1} \in P_1(K)$ on the one hand, and since $f(\xi)$ is invariant by HCT-R interpolation on the other hand, we find from (4.2-52)

$$(4.2\text{-}56)\quad \left\{\begin{aligned} &\sum_{i=1}^{3} A_1(\Sigma_i)v_{h1,1}(\Sigma_i)\ \{(\xi^1_{i+1}-\xi^1_i)(p^1_{j,i,i+1})_{,\alpha\beta} + \\ &\qquad + (\xi^1_{i-1}-\xi^1_i)(p^1_{j,i,i-1})_{,\alpha\beta}\} = - \delta^1_\alpha\, \delta^1_\beta\, v_{h1,1}(\xi) \end{aligned}\right.$$

Then, by virtue of (4.2-55) and (4.2-56), we obtain :

$$
(4.2\text{-}57)\quad \left\{
\begin{aligned}
&\sum_{i=1}^{3} A_1(\Sigma_i)v_{h1}(\Sigma_i)(p^{o}_{j,i})_{,\alpha\beta} + \sum_{i=1}^{3}\{\tfrac{1}{2}\, e_{1\nu}\, A_1(\Sigma_i)[v_{h1,2}(\xi) - v_{h2,1}(\xi)] - \\
&\qquad - \delta^1_\nu v_{h1}(\Sigma_i)\}\ \{(\xi^\nu_{i+1}-\xi^\nu_i)(p^1_{j,i,i+1})_{,\alpha\beta} + \\
&\qquad + (\xi^\nu_{i-1}-\xi^\nu_i)(p^1_{j,i,i-1})_{,\alpha\beta}\} = \\
&= - \delta^1_\alpha\, \delta^1_\beta\, v_{h1,1}(\xi) - (\delta^1_\alpha\, \delta^2_\beta + \delta^2_\alpha\, \delta^1_\beta)v_{h1,2}(\xi) - \\
&\qquad - \frac{1}{2}\sum_{i=1}^{3} A_1(\Sigma_i)\{(\xi^2_{i+1}-\xi^2_i)(p^1_{j,i,i+1})_{,\alpha\beta} + \\
&\qquad + (\xi^2_{i-1}-\xi^2_i)(p^1_{j,i,i-1})_{,\alpha\beta}\}\ (v_{h1,2}(\xi) + v_{h2,1}(\xi))
\end{aligned}
\right.
$$

It remains to study the last term of above relation.

For clarity, we denote

$$\xi^1_2 - \xi^1_1 = h_1 \quad , \quad \xi^2_3 - \xi^2_1 = h_2 \quad , \quad \xi^2_1 - \xi^2_2 = h_3 \quad .$$

Then, we obtain

$$
(4.2\text{-}58)\quad \left\{
\begin{aligned}
&\sum_{i=1}^{3} A_1(\Sigma_i)\ \{(\xi^2_{i+1}-\xi^2_i)(p^1_{j,i,i+1})_{,\alpha\beta} + (\xi^2_{i-1}-\xi^2_i)(p^1_{j,i,i-1})_{,\alpha\beta}\} = \\
&\qquad = - \frac{1}{2}\ \{h_1h_3[(p^1_{j,1,2})_{,\alpha\beta} + (p^1_{j,2,1})_{,\alpha\beta}] + \\
&\qquad\quad + h_1(h_2 + h_3)[(p^1_{j,2,3})_{,\alpha\beta} + (p^1_{j,3,2})_{,\alpha\beta}] + \\
&\qquad\quad + h_1h_2\ [(p^1_{j,3,1})_{,\alpha\beta} - (p^1_{j,1,3})_{,\alpha\beta}]\}
\end{aligned}
\right.
$$

If we consider the function $g(\xi) = (\xi^1-\xi^1_1)(\xi^2-\xi^2_2)$, we have :

$$
\left\{
\begin{aligned}
&g(\Sigma_i) = 0 \\
&g_{,1}(\Sigma_1) = h_3 \qquad\qquad\qquad \left\{\begin{aligned} &g_{,2}(\Sigma_1) = 0 \\ &g_{,2}(\Sigma_2) = h_1 \\ &g_{,2}(\Sigma_3) = 0 \end{aligned}\right. \\
&g_{,1}(\Sigma_2) = 0 \\
&g_{,1}(\Sigma_3) = h_2 + h_3 \\
&\qquad\qquad g_{,\alpha\beta}(\xi) = \delta^1_\alpha\, \delta^2_\beta + \delta^2_\alpha\, \delta^1_\beta
\end{aligned}
\right.
$$

and thus from (4.2-52)

$$(4.2\text{-}59)\quad \left\{\begin{aligned} & h_1h_3[(p^1_{j,1,2})_{,\alpha\beta} + (p^1_{j,2,1})_{,\alpha\beta}] + h_1(h_2+h_3)[(p^1_{j,2,3})_{,\alpha\beta} + \\ & \qquad + (p^1_{j,3,2})_{,\alpha\beta}] = \delta^1_\alpha\,\delta^2_\beta + \delta^2_\alpha\,\delta^1_\beta \ . \end{aligned}\right.$$

By virtue of (4.2-57), (4.2-58) and (4.2-59), we find for any $\xi \in K_j$

$$(4.2\text{-}60)\quad \left\{\begin{aligned} & \sum_{i=1}^{3} A_1(\Sigma_i)v_{h1}(\Sigma_i)(p^o_{j,i})_{,\alpha\beta} + \sum_{i=1}^{3}\{\tfrac{1}{2}\, e_{1\nu}A_1(\Sigma_i)[v_{h1,2}(\xi)-v_{h2,1}(\xi)] - \\ & \qquad - \delta^1_\nu v_{h1}(\Sigma_i)\}\ \{(\xi^\nu_{i+1}-\xi^\nu_i)(p^1_{j,i,i+1})_{,\alpha\beta} + \\ & \qquad + (\xi^\nu_{i-1}-\xi^\nu_i)(p^1_{j,i,i-1})_{,\alpha\beta}\} = \\ & = -\,\delta^1_\alpha\delta^1_\beta v_{h1,1}(\xi) - \tfrac{1}{4}(\delta^1_\alpha\delta^2_\beta + \delta^2_\alpha\delta^1_\beta)\{3v_{h1,2}(\xi)-v_{h2,1}(\xi)\} + h_1h_2\Delta_{\alpha\beta}(\vec{v}_h), \end{aligned}\right.$$

where

$$(4.2\text{-}61)\qquad \Delta_{\alpha\beta}(\vec{v}_h) = \tfrac{1}{4}\,\{(p^1_{j,3,1})_{,\alpha\beta} - (p^1_{j,1,3})_{,\alpha\beta}\}\ \{v_{h1,2}(\xi) + v_{h2,1}(\xi)\}$$

Upon combining (4.2-49),(4.2-50),(4.2-52),(4.2-53),(4.2-54) and (4.2-60) we deduce by using (4.2-26) the existence of a constant C, independent of h, such that on every triangle K

$$(4.2\text{-}62)\quad \left\{\begin{aligned} & \tilde{\rho}_{h\alpha\beta}(\tilde{\vec{v}}_h) = \rho_{\alpha\beta}(\vec{v}_h) + \tfrac{1}{R}\,h_1h_2\Delta_{\alpha\beta}(\vec{v}_h) + \\ & \qquad + O(h)\ [v_{h\varepsilon,\eta} + \sum_{k=1}^{3}\{C^j_{\alpha\beta}\ v_{hj}(\Sigma_k) + C^\varepsilon_{\alpha\beta}\ v_{h3,\varepsilon}(\Sigma_k) + C^\varepsilon_{\alpha\beta}\ v_{h3,\varepsilon}(\bar{\xi}_k)] \end{aligned}\right.$$

From (4.2-21), we derive

$$a^{\alpha\beta}_h = a^{\alpha\beta}(\xi) + O(h^2)$$

so that relations (4.2-26), (4.2-27) and (4.2-62) give

$$(4.2\text{-}63)\quad \left\{\begin{aligned} & \tilde{\rho}^{\ \alpha}_{h\beta}(\tilde{\vec{v}}_h) = \rho^\alpha_\beta(\vec{v}_h) + \tfrac{1}{R}\,h_1h_2\ a^{\alpha\lambda}(\xi)\Delta_{\lambda\beta}(\vec{v}_h) + \\ & \qquad + O(h)\ [v_{h\varepsilon,\eta} + \sum_{k=1}^{3}\{C^j_{\alpha\beta}\ v_{hj}(\Sigma_k) + C^\varepsilon_{\alpha\beta}\ v_{h3,\varepsilon}(\Sigma_k) + C^\varepsilon_{\alpha\beta}\ v_{h3,\varepsilon}(\bar{\xi}_k)] \end{aligned}\right.$$

Step 6 : Obtention of estimate (4.2-29)

From finite expansion (4.2-63) and by taking into account that $\mathrm{meas}(K) = O(h^2)$, we derive for any triangle K of $\mathcal{C}_h$ the existence of a constant C independent of h such that

$$(4.2\text{-}64)\quad \left\{ \begin{aligned} & \left|\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h) - \rho^{\alpha}_{\beta}(\vec{v}_h) - \frac{1}{R}\, h_1 h_2 a^{\alpha\lambda}(\xi)\Delta_{\lambda\beta}(\vec{v}_h)\right|_{0,K} \leq \\ & \leq \; Ch^2\{|v_{hj}|^2_{0,\infty,K} + |v_{hj}|^2_{1,\infty,K}\}^{1/2} \quad . \end{aligned}\right.$$

By using Lemma 4.2-1 and results of interpolation theory in Sobolev spaces, - CIARLET [1, Theorems 3.1-2 and 3.1-3] , we deduce

$$|v_{hj}|_{p,\infty,K} \leq Ch^{-1}|v_{hj}|_{p,K} \quad , \quad p = 0,1 \quad ,$$

inequality which, combined with (4.2-64), leads to

$$(4.2\text{-}65)\quad \left\{ \begin{aligned} & \left|\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h) - \rho^{\alpha}_{\beta}(\vec{v}_h) - \frac{1}{R}\, h_1 h_2 a^{\alpha\lambda}(\xi)\Delta_{\lambda\beta}(\vec{v}_h)\right|_{0,K} \leq \\ & \leq Ch\{\|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{2,K}\}^{1/2} \quad . \end{aligned}\right.$$

For clarity, we denote

$$\left\{ \begin{aligned} & p_{\alpha\beta} = (p^1_{j,3,1})_{,\alpha\beta} - (p^1_{j,1,3})_{,\alpha\beta} \\ & w_h = \frac{1}{4}\,(v_{h1,2} + v_{h2,1}) \end{aligned}\right.$$

so that we obtain with (4.2-61)

$$\Delta_{\alpha\beta}(\vec{v}_h) = p_{\alpha\beta} w_h \quad .$$

By using results of interpolation theory in Sobolev spaces

$$|\Delta_{\alpha\beta}(\vec{v}_h)|_{0,K} = |p_{\alpha\beta}w_h|_{0,K} \leq Ch\ |\hat{p}_{\alpha\beta}\hat{w}_h|_{0,\hat{K}}$$

and thus

$$(4.2\text{-}66) \qquad |\Delta_{\alpha\beta}(\vec{v}_h)|_{0,K} \leq Ch\ |\hat{p}_{\alpha\beta}|_{0,\infty,\hat{K}}\ |\hat{w}_h|_{0,\hat{K}} \quad ,$$

where $\hat{K}$ is a reference triangle (for example, the triangle of vertices $\hat{a}_1 = (0,0)$, $\hat{a}_2 = (0,1)$, $\hat{a}_3 = (1,0)$).

From Lemma 4.2-1, we find

$$(4.2\text{-}67) \qquad |\hat{p}_{\alpha\beta}|_{0,\infty,\hat{K}} \leq C|\hat{p}_{\alpha\beta}|_{0,\hat{K}} \leq C\{|\hat{p}^1_{j,1,3}|_{2,\hat{K}} + |\hat{p}^1_{j,3,1}|_{2,\hat{K}}\} \quad .$$

By using again results of interpolation theory, we find

$$(4.2\text{-}68) \quad \left\{ \begin{array}{l} |\hat{w}_h|_{0,\hat{K}} \leq Ch^{-1}\ |w_h|_{0,K} \\ |\hat{p}^1_{j,1,3}|_{2,\hat{K}} \leq Ch\ |p^1_{j,1,3}|_{2,K} \ \text{(and a similar inequality for } |\hat{p}^1_{j,3,1}|_{2,\hat{K}}) \end{array} \right.$$

Upon combining (4.2-67) and (4.2-68) with (4.2-66), we obtain

$$(4.2\text{-}69) \qquad |\Delta_{\alpha\beta}(\vec{v}_h)|_{0,K} \leq Ch\ |w_h|_{0,K}\ \{|p^1_{j,1,3}|_{2,K} + |p^1_{j,3,1}|_{2,K}\} \quad .$$

Then, by taking into account that meas(K) = $O(h^2)$, we have

$$\left\{ \begin{array}{l} |p^1_{j,1,3}|_{2,K} \leq Ch^{-1} \\ |p^1_{j,3,1}|_{2,K} \leq Ch^{-1} \end{array} \right.$$

so that inequality (4.2-69) gives

$$|\Delta_{\alpha\beta}(\vec{v}_h)|_{0,K} \leq C_{\alpha\beta}|w_h|_{0,K}$$

and thus, since the mappings $a^{\alpha\beta}$ are constant on $\bar{K}$, we derive

$$(4.2\text{-}70) \qquad |a^{\alpha\lambda}(\xi)\Delta_{\lambda\beta}(\vec{v}_h)|_{0,K} \leq C\{\|v_{h1}\|_{1,K} + \|v_{h2}\|_{1,K}\}$$

Since we have

$$\begin{cases} |\tilde{\rho}_{h\beta}^{\ \alpha}(\vec{\tilde{v}}_h) - \rho_\beta^\alpha(\vec{v}_h)|_{0,K} \leq \\ \quad \leq |\tilde{\rho}_{h\beta}^{\ \alpha}(\vec{\tilde{v}}_h) - \rho_\beta^\alpha(\vec{v}_h) - h_1 h_2 a^{\alpha\lambda}\Delta_{\lambda\beta}(\vec{v}_h)|_{0,K} + h_1 h_2 |a^{\alpha\lambda}\Delta_{\lambda\beta}(\vec{v}_h)|_{0,K} \end{cases}$$

by combining estimates (4.2-65) and (4.2-70), we finally deduce estimate (4.2-29).

□

Now, we are able to prove the following theorem giving the searched estimate of $|a(\vec{v}_h,\vec{w}_h) - \tilde{a}_h(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h)|$, or, equivalently with (3.3-5), of $|a(\vec{v}_h,\vec{w}_h) - a_h(\vec{v}_h,\vec{w}_h)|$.

<u>Theorem 4.2-3</u> : <u>For any regular triangulation</u> $\mathcal{T}_h$ <u>of</u> $\bar{\Omega}$ <u>satisfying assumption</u> (4.2-28), <u>there exists a constant</u> C, <u>independent of</u> h, <u>such that, for any</u> $\vec{v}_h,\vec{w}_h \in \vec{X}_h$,

$$(4.2\text{-}71) \qquad |a(\vec{v}_h,\vec{w}_h) - a_h(\vec{v}_h,\vec{w}_h)| \leq Ch\|\vec{v}_h\|\ \|\vec{w}_h\| \ .$$

<u>Proof</u> : By virtue of (4.2-1) and using (4.2-4), it suffices to estimate $|ER_{K_1}(\vec{v}_h,\vec{w}_h)|$, the analysis being the same for the other terms $ER_{K_i}(\vec{v}_h,\vec{w}_h)$, $2 \leq i \leq 4$. We have

$$(4.2\text{-}72) \quad \begin{cases} \int_K [\gamma_\beta^\alpha(\vec{v}_h)\gamma_\alpha^\beta(\vec{w}_h)\sqrt{a} - \tilde{\gamma}_{h\beta}^{\ \alpha}(\vec{\tilde{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\vec{\tilde{w}}_h)\sqrt{a_h}]\, d\xi^1\, d\xi^2 = \\ \qquad = \int_K (\sqrt{a} - \sqrt{a_h})\gamma_\beta^\alpha(\vec{v}_h)\gamma_\alpha^\beta(\vec{w}_h)\, d\xi^1\, d\xi^2 + \\ \qquad + \int_K \sqrt{a_h}\, [\gamma_\beta^\alpha(\vec{v}_h)\gamma_\alpha^\beta(\vec{w}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\vec{\tilde{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\vec{\tilde{w}}_h)]\, d\xi^1\, d\xi^2 \ . \end{cases}$$

Applying (4.2-42), we obtain

$$(4.2\text{-}73) \qquad |\sqrt{a} - \sqrt{a}_h|_{0,\infty,K} \le Ch^2 ,$$

where C is a constant independent of h.

For the first term appearing in the right part of (4.2-72), we have

$$\left\{ \begin{aligned} & \left|\int_K (\sqrt{a} - \sqrt{a}_h)\gamma^\alpha_\beta(\vec{v}_h)\gamma^\beta_\alpha(\vec{w}_h)\, d\xi^1\, d\xi^2\right| \le \\ & \qquad \le |\sqrt{a} - \sqrt{a}_h|_{0,\infty,K}\, |\gamma^\alpha_\beta(\vec{v}_h)|_{0,K}\, |\gamma^\beta_\alpha(\vec{w}_h)|_{0,K} \end{aligned} \right.$$

Since the mappings $a_{\alpha\beta}$, $a^{\alpha\beta}$, a , $b_{\alpha\beta}$, b^α_β are constant on $\bar{\Omega}$, we deduce with estimates (4.2-73)

$$(4.2\text{-}74) \qquad \left|\int_K (\sqrt{a} - \sqrt{a}_h)\gamma^\alpha_\beta(\vec{v}_h)\gamma^\beta_\alpha(\vec{w}_h)\, d\xi^1\, d\xi^2\right| \le Ch^2\|\vec{v}_h\|_K\, \|\vec{w}_h\|_K .$$

For the second term appearing in (4.2-72), we write

$$(4.2\text{-}75) \qquad \left\{ \begin{aligned} & \left|\int_K \sqrt{a}_h\, [\gamma^\alpha_\beta(\vec{v}_h)\gamma^\beta_\alpha(\vec{w}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)]\, d\xi^1\, d\xi^2\right| \le \\ & \quad \le \left|\int_K (\sqrt{a}_h - \sqrt{a})\, [\gamma^\alpha_\beta(\vec{v}_h)\gamma^\beta_\alpha(\vec{w}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)]\, d\xi^1\, d\xi^2\right| + \\ & \quad + \left|\int_K \sqrt{a}\, [\gamma^\alpha_\beta(\vec{v}_h)\gamma^\beta_\alpha(\vec{w}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)]\, d\xi^1\, d\xi^2\right|. \end{aligned} \right.$$

But,

$$\left\{ \begin{aligned} & \left|\int_K (\sqrt{a} - \sqrt{a}_h)\, [\gamma^\alpha_\beta(\vec{v}_h)\gamma^\beta_\alpha(\vec{w}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)]\, d\xi^1\, d\xi^2\right| \le \\ & \le |\sqrt{a} - \sqrt{a}_h|_{0,\infty,K}\, \{|\gamma^\alpha_\beta(\vec{v}_h)|_{0,K}|\gamma^\beta_\alpha(\vec{w}_h) - \tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)|_{0,K} + \\ & + |\gamma^\beta_\alpha(\vec{w}_h)|_{0,K}|\gamma^\alpha_\beta(\vec{v}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)|_{0,K} + \\ & + |\gamma^\alpha_\beta(\vec{v}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)|_{0,K}|\gamma^\beta_\alpha(\vec{w}_h) - \tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)|_{0,K}\} , \end{aligned} \right.$$

and since the mappings $a_{\alpha\beta}$, $a^{\alpha\beta}$, a , $b_{\alpha\beta}$, b^{α}_{β} are constants on $\bar{\Omega}$, we obtain with (4.2-73) and (4.2-11)

$$(4.2\text{-}76)\quad \left\{ \begin{aligned} & \left| \int_K (\sqrt{a} - \sqrt{a_h}) [\gamma^{\alpha}_{\beta}(\vec{v}_h)\gamma^{\beta}_{\alpha}(\vec{w}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)]\, d\xi^1\, d\xi^2 \right| \le \\ & \qquad \le \ Ch^3 \, \|\vec{v}_h\|_K \, \|\vec{w}_h\|_K \ . \end{aligned} \right.$$

In the same way,

$$(4.2\text{-}77)\quad \left\{ \begin{aligned} & \left| \int_K \sqrt{a}\ [\gamma^{\alpha}_{\beta}(\vec{v}_h)\gamma^{\beta}_{\alpha}(\vec{w}_h) - \tilde{\gamma}_{h\beta}^{\ \alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}_{h\alpha}^{\ \beta}(\tilde{\vec{w}}_h)]\, d\xi^1\, d\xi^2 \right| \le \\ & \qquad \le \ Ch \, \|\vec{v}_h\|_K \, \|\vec{w}_h\|_K \ . \end{aligned} \right.$$

Upon combining (4.2-5), (4.2-72) and (4.2-74) to (4.2-77), we finally obtain :

$$|ER_{K_1}(\vec{v}_h, \vec{w}_h)| \le Ch \, \|\vec{v}_h\|_K \, \|\vec{w}_h\|_K \ .$$

With a similar analysis, we easily obtain

$$|ER_{K_i}(\vec{v}_h, \vec{w}_h)| \le Ch \, \|\vec{v}_h\|_K \, \|\vec{w}_h\|_K \ , \quad \forall i \ , \ 1 \le i \le 4 \quad ,$$

and the proof of estimate (4.2-71) is achieved by summation over all the triangles K of $\mathcal{T}_h$.

□

4.3 - Uniform $\vec{V}_h$-ellipticity of the bilinear form $a_h(.,.)$ given in (3.3-5) :

As a consequence of Theorem 4.2-3, we deduce the following result :

Theorem 4.3-1 : Under the assumption (4.2-28) and for h sufficiently small, the bilinear form (3.3-5) is uniformly $\vec{V}_h$-elliptic.

<u>Proof</u> : For any $\vec{v}_h \in \vec{V}_h$, we have

$$a_h(\vec{v}_h,\vec{v}_h) = a(\vec{v}_h,\vec{v}_h) + (a_h(\vec{v}_h,\vec{v}_h) - a(\vec{v}_h,\vec{v}_h)) \ .$$

From Theorem 1.3-1, we know that the bilinear form $a(.,.)$ is $\vec{V}$-elliptic. Using the inclusion $\vec{V}_h \subset \vec{V}$, we deduce that there exists a constant $\alpha > 0$, independent of h, such that

$$\forall \vec{v}_h \in \vec{V}_h \ , \quad \alpha \, \|\vec{v}_h\|^2 \leq a(\vec{v}_h,\vec{v}_h) \ .$$

Combining this inequality with estimate (4.2-71), we obtain

$$\forall \vec{v}_h \in \vec{V}_h \ , \quad a_h(\vec{v}_h,\vec{v}_h) \geq (\alpha - Ch) \, \|\vec{v}_h\|^2 \ .$$

Then, for h sufficiently small (more precisely, for h such as $h \leq \frac{\alpha}{2C}$), we get

$$\forall \vec{v}_h \in \vec{V}_h \ , \ a_h(\vec{v}_h,\vec{v}_h) \geq \beta \, \|\vec{v}_h\|^2 \quad , \qquad \beta = \frac{\alpha}{2} \ .$$

□

Now, it remains to estimate the consistency between the linear forms $f(.)$ and $f_h(.)$ respectively defined in (1.2-4) and (3.3-3).

4.4 - <u>Estimate of the consistency error</u> $|f(\vec{v}_h) - f_h(\vec{v}_h)|$

Similarly to Theorem 4.2-3, we obtain the following result :

<u>Theorem 4.4-1</u> : <u>For any regular triangulation</u> $\mathcal{T}_h$ <u>of</u> $\bar{\Omega}$ <u>and for any components</u> $p^i \in W^{1,q}(\Omega)$, $q \in \mathbb{R}$, $q \geq 2$, <u>defined in</u> (1.2-4) <u>there exists a constant</u> C, <u>independent of</u> h, <u>such that, for any</u> $\vec{w}_h \in \vec{V}_h$,

$$(4.4\text{-}1) \qquad |f(\vec{w}_h) - f_h(\vec{w}_h)| \leq Ch \, \|\vec{p}\|_{1,q,\Omega} \, \|\vec{w}_h\| \ .$$

Proof : First, we can observe that the inclusion with continuous injection $W^{1,q}(\Omega) \hookrightarrow \mathscr{C}^o(\Omega)$ gives sense to the expression (3.3-3).

Next, observe that this definition (3.3-3) is obtained from definition (1.2-4) by using the following numerical integration scheme over the reference triangle $\hat{K}$ (for example, the triangle with vertices $\hat{a}_1 = (0,0)$, $\hat{a}_2 = (0,1)$, $\hat{a}_3 = (1,0)$) :

$$\int_{\hat{K}} \hat{\phi}(\hat{\xi})\, d\hat{\xi} \sim \frac{\text{meas}(\hat{K})}{3} \sum_{i=1}^{3} \hat{\phi}(\hat{a}_i) \ .$$

Since this scheme is in particular exact for constants, the estimate (4.4-1) follows from results of BERNADOU ([1, part III, Theorem 4.4-2] or [2, Theorem 4.2]).

□

4.5 - Convergence and error estimate

Now, we are able to prove the following convergence result :

Theorem 4.5-1 : For any regular triangulation $\mathscr{T}_h$ of $\bar{\Omega}$ satisfying assumption (4.2-28) and for h sufficiently small, the discrete problem (3.3-4) has one and only one solution $\vec{u}_h \in \vec{V}_h$.

Moreover, for any components of the load $p^i \in W^{1,q}(\Omega)$, $q \in \mathbb{R}$, $q \geq 2$, defined in (1.2-4), there exists a constant C, independent of h, such that, for h small enough,

$$\text{(4.5-1)} \qquad \|\vec{u} - \vec{u}_h\| \leq Ch \ ,$$

where $\vec{u}$ denotes the unique solution of the continuous problem (1.2-5).

<u>Proof</u> : Denoting by Π_h the $\vec{V}_h$-interpolation operator, one has

$$\left\{\begin{aligned} &\inf_{\vec{v}_h\in\vec{V}_h}\{\|\vec{u}-\vec{v}_h\| + \sup_{\vec{w}_h\in\vec{V}_h}\frac{|a(\vec{v}_h,\vec{w}_h) - a_h(\vec{v}_h,\vec{w}_h)|}{\|\vec{w}_h\|}\} \leq \\ &\leq \|\vec{u}-\Pi_h\vec{u}\| + \sup_{\vec{w}_h\in\vec{V}_h}\frac{|a(\Pi_h\vec{u},\vec{w}_h) - a_h(\Pi_h\vec{u},\vec{w}_h)|}{\|\vec{w}_h\|} .\end{aligned}\right.$$

Thus, by using Theorems 2.2-2 and 4.2-3, we obtain

$$(4.5\text{-}2)\quad\left\{\begin{aligned} &\inf_{\vec{v}_h\in\vec{V}_h}\{\|\vec{u}-\vec{v}_h\| + \sup_{\vec{w}_h\in\vec{V}_h}\frac{|a(\vec{v}_h,\vec{w}_h) - a_h(\vec{v}_h,\vec{w}_h)|}{\|\vec{w}_h\|}\} \leq \\ &\leq Ch\,\{(|u_1|^2_{2,\Omega} + |u_2|^2_{2,\Omega} + |u_3|^2_{3,\Omega})^{1/2} + \|\vec{u}\|\} .\end{aligned}\right.$$

By virtue of Theorem 4.3-1, we can apply the results of Theorem 4.1-1 so that we deduce by combining (4.5-2) with Theorem 4.4-1 :

$$\|\vec{u}-\vec{u}_h\| \leq Ch\,\{(|u_1|^2_{2,\Omega} + |u_2|^2_{2,\Omega} + |u_3|^2_{3,\Omega})^{1/2} + \|\vec{u}\| + \|\vec{p}\|_{1,q,\Omega}\}$$

and estimate (4.5-1) follows.

□

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